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Dual Field Theories In $(d-1)+1$ Emergent Spacetimes From A Unifying Field Theory In $d+2$ Spacetime[1]

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Abstract

According to Two-Time Physics, there is more to space-time than can be garnered with the ordinary formulation of physics. Two-Time Physics has shown that the Standard Model of Particles and Forces is successfully reproduced by a two-time field theory in 4 space and 2 time dimensions projected as a holographic image on an emergent spacetime in 3+1 dimensions. Among the successes of this approach is the resolution of the strong CP problem of QCD as an outcome of the restrictions imposed by the higher symmetry structures in 4+2 dimensions. In this paper we launch a program to construct the duals of the Standard Model as other holographic images of the same 4+2 dimensional theory on a variety of emergent spacetimes in 3+1 dimensions. These dual field theories are obtained as a family of gauge choices in the master 4+2 field theory. In the present paper we deal with some of the simpler gauge choices which lead to interacting Klein-Gordon field theories for the conformal scalar with a predicted SO(d,2) symmetry in a variety of interesting curved spacetimes in (d-1)+1 dimensions. More challenging and more interesting gauge choices (including some that relate to mass) which are left to future work are also outlined. Through this approach we discover a new realm of previously unexplored dualities and hidden symmetries that exist both in the macroscopic and microscopic worlds, at the classical and quantum levels. Such phenomena predicted by 2T-physics can in principle be confirmed both by theory and experiment. 1T-physics can be used to analyze the predictions but in most instances gives no clue that the predicted phenomena exist in the first place. This point of view suggests a new paradigm for the construction of a fundamental theory that is likely to impact on the quest for unification.

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Contents

I. 2T-PHYSICS VERSUS 1T-PHYSICS

Evidence has been mounting that the ordinary formulation of physics, in a space-time with three space and one time dimensions (1T-physics), is insufficient to describe certain aspects of our world, just like shadows on walls alone are insufficient to capture the true essence of an object in a three dimensional room. Two-Time Physics (2T-physics) [1]-[19] has revealed that our physical world in three space and one time dimensions is like a holographic shadow of a highly symmetric universe in four space and two time dimensions. Amazingly, the fundamental theory in Physics that agrees exquisitely with experiment as we know it today, the Standard Model of Particles and Forces in 3+1 dimensions, is reproduced, and to boot its "strong CP problem" is solved, as a holographic "shadow" of a field theory in 4+2 dimensions [17].

A surprising outcome of the 4+2 point of view (more generally $d+2$) is that the master theory has many 3+1 dimensional holographic "shadows" (more generally $(d-1)+1$), which are distinguishable from the point of view of 1T-physics. Thus, the 2T-physics formulation predicts that the usual Standard Model has close relatives, i.e. other "shadows", that look like different field theories, but yet represent the same theory in 4+2 dimensions, so that they are dual to the Standard Model. A variety of "shadows" are predicted to appear with different dynamics in 1T-physics (i.e. different Hamiltonians, or different field equations), to have hidden symmetries that signal an extra space and an extra time, and to have an infinite set of calculable and measurable relationships among them, akin to dualities.

Since these "shadows" belong to the same theory, a new type of unification of different field theories emerges through higher dimensions that includes one extra space and one extra time dimensions. This type of unification is very different from the Kaluza-Klein idea because there are no Kaluza-Klein type remnant degrees of freedom from the extra 1+1 spacetime, but instead there is a family of dual field theories.

The extra 1+1 dimensions required by 2T-physics are different than Kaluza-Klein type of extra dimensions, however both types of extra dimensions could coexist in the same theory since their roles are not incompatible with each other.

The reader that is familiar with M theory may recognize some parallels to dualities in M theory. In this paper we will make the case that there are conceptually similar dualities at the level of field theory that include the Standard Model.

In 2T-physics the concepts outlined above are realized in explicit dynamical theories which apply to both macroscopic and microscopic physics, at the classical and quantum levels. The dualities are related to a gauge symmetry which treats the extra 1+1 space-time as gauge degrees of freedom embedded in $d+2$ dimensions.

Such properties have previously been exhibited with explicit examples in the worldline formalism for particles [1], including spin [2][19], supersymmetry [11][8][10][12], and interactions with background fields [6][7]. What ties the dual images together is the higher dimensional "master theory" in $d+2$ dimensions which is subject to certain gauge symmetries. As the master theory is changed (and there is an infinite set of possibilities in the worldline formalism), new shadows are generated with new duality relationships and new hidden symmetries.

The underlying reason for such striking properties cannot be found in 1T-physics in $(d-1)+1$ dimensions, but is explained in 2T-physics [1] as being due to a fundamental $\mathrm{Sp}(2,R)$ gauge symmetry which acts in *phase space* $\left(X^M, P_M\right)$ and makes position and momentum indistinguishable at any instant. This gauge symmetry can be implemented only if the target spacetime includes one extra space and one extra time dimensions, thus showing that the unification relies on a spacetime in $d+2$ dimensions demanded by the $\mathrm{Sp}(2,R)$ gauge symmetry.

In general the $d+2$ dimensional spacetime is not necessarily flat, so background fields of all spins are permitted provided they satisfy some restrictions in $d+2$ dimensions [6][7].

The evidence of the $d+2$ dimensional world in the form of hidden symmetries and dualities can be found both at the macroscopic and microscopic scales, and such predictions of Two-Time Physics can be tested through both theory and experiment. The existence of such relations and hidden symmetries, never suspected in 1T-physics, shows that the higher space in $d+2$ dimensions is not just formalism that could be avoided. 1T-physics can be used to verify and interpret the predictions of 2T-physics, but it is not equipped to come up with the predictions in the first place, unless one stumbles into some of them occasionally, such as $\mathrm{SO}(4,2)$ conformal symmetry of massless systems. The lessons from 2T-physics so far makes it evident that the ordinary 1T-physics formulation of Nature is insufficient to provide the explanation or even the existence of the many unifying facts revealed through 2T-physics.

To settle the physical and philosophical interpretation of 2T-physics will probably require

much discussion in the future. For the time being we are content to use the 2T formalism at least as an approach that provides new mathematical tools and new physical insights for understanding our universe. In this paper we launch a program to exploit such properties of 2T-physics in the context of field theory, to develop new analytical methods that would be useful in their own right directly in 1T-physics, and that will help us understand the deeper implications of 2T-physics.

II. NEW EMERGENT PRINCIPLES IN FIELD THEORY

As mentioned above, two-time physics [1] is based on gauging the symplectic transformations of phase space $\left(X^M, P_M\right)$, $\mathrm{Sp}(2,\mathbb{R})$. One of the fundamental results of this new gauge principle is that, in order to be nontrivial, it requires the theory to be formulated in a spacetime having at least two times. While the choice of exactly two timelike dimensions results in a coherent theory, investigations of alternatives with more than two times have been done [20]. So far, these appear to rule out such possibilities and seem to confirm the special status of 2T-physics.

The theory was first formulated in the worldline formalism in which the $\mathrm{Sp}(2,\mathbb{R})$ gauge symmetry allows the elimination of one spacelike and two timelike degrees of freedom. When quantized, the theory is seen to be completely free of ghosts, thus confirming the viability of the theory. In this paper our investigation will be at the level of field theory, rather than worldline theory. The connection between the two is that the field $\Phi\left(X\right)$ corresponds to the first quantized wavefunction $\Phi\left(X\right) = \langle X|\Phi\rangle$, and in the field theory context we also include field interactions.

Although an infinite set of $\mathrm{Sp}(2,R)$ gauge invariant theories exist for describing particles on the worldline moving in arbitrary backgrounds [6][7], so far most of the investigations of 2T-physics have concentrated on the free particle in flat $d+2$ dimensions with an $\mathrm{SO}(d,2)$ global symmetry. This simplest flat background in $d+2$ dimensions, which leads to a rich set of backgrounds and dualities in $(d-1)+1$ dimensions, as discussed in section III, will also be adopted in the context of 2T-field theory in this paper.

An important general feature of 2T physics on the worldline is that the potentially infinite variety of gauge choices leads to an equally infinite family of lower dimensional systems. All the possible gauge choices have not been classified. A list of the known gauge choices of the

simplest theory on the worldline is provided in section III. It should be noted that, while the master worldline theory is the free particle in flat spacetime in $d+2$ dimensions subject to the Sp$(2,R)$ constraints, the emergent $(d-1)+1$ dimensional worldline systems include both free and interacting particles, in flat or curved spaces, with or without mass. The parameters that describe mass, interaction, curvature, etc. emerge from the extra dimensions as moduli that parameterize the gauge fixed phase space.

The issue of gauge choice is related to the question of which $(d-1)+1$ dimensions is embedded in $d+2$ dimensions. The different embeddings of *phase space* (not just space) in $(d-1)+1$ dimensions into phase space in $d+2$ dimensions potentially creates a huge variety of dual field theories. It is expected that the investigation of these duals in the context of field theory could lead to further insight into nonperturbative aspects of the theory (such as QCD) and might also provide new calculational tools.

While these systems look very different in the particle worldline theory (different Hamiltonians), the higher dimensional theory actually proves that these are all dual to one another and establishes that dualities must connect them - a duality which would have been hard to show otherwise. In particular, all these systems have a hidden global SO$(d,2)$ symmetry which is a manifest global symmetry of the parent theory in flat space. The different realizations of this symmetry in the lower dimensions are often highly nonlinear and, again, would have been hard to find directly. The gauge choices leading to different 1T-physics systems have their counterparts in field theory, and the construction of the corresponding dual field theories, and their hidden SO$(d,2)$ global symmetry, is the focus of our investigation.

The worldline gauge choices provided in section III will be adopted for similar gauge choices in 2T-physics field theory that will be discussed in this paper. These will lead to the *emergent field theories in various spacetimes.* In addition to the interaction features and parameters of the emergent spacetimes mentioned above (mass, interaction, curvature, etc.), the 2T-physics field theory adds local field interactions in $d+2$ dimensions. Thus a given emergent field theory system in $(d-1)+1$ dimensions, which is a member of the duality family, receives contributions to its interaction structure both from the embedding in $d+2$ dimensions and from the field interactions directly in $d+2$ dimensions.

First quantization of the emerging worldline systems in specific gauges can be tricky in terms of the respective phase spaces, since in first quantization ordering issues of nonlinear expressions must be taken care of in order to preserve the SO$(d,2)$ symmetry at the quantum

level. A recent success that overcomes this issue automatically was the formulation of the field theory approach directly at the two-time level, with its own new gauge symmetry that is related to the underlying Sp$(2,R)$ [17]. Gauge fixing of the field theory itself can then be performed in this 2T field theory and it has been shown (with further evidence in the present paper) that the resulting lower-dimensional field equations agree with the wavefunction equations obtained by first quantization of the gauge fixed system in the worldline formalism [3][4]. 2T field theory has the advantage that any potential quantum ordering ambiguities of the first quantized theory are automatically resolved.

More importantly, the field interactions introduced in flat $d+2$ dimensions are consistent with field interactions in flat or curved $(d-1)+1$ dimensions, but restricts to some extent the possible interactions in the lower dimension. Interestingly, for the $4+2$ case, the emergent theory in flat $3+1$ dimensions allows most renormalizable interactions that correspond to dimension 4 operators. The restriction that emerge on those interactions are quite interesting:

- Dimensionful parameters such as masses are not permitted by the gauge symmetries in the 4+2 theory, so in the emergent $3+1$ theory masses can only emerge either from the extra dimensions as outlined above, or from spontaneous breakdown as outlined in [17]. This mass feature may also be helpful for a resolution of the gauge hierarchy problem with a mechanism that is different from supersymmetry, but this issue remains to be better understood in the quantum analysis of the theory.

- Furthermore *renormalizable* terms of the form $Tr\,(F_{\mu\nu}F_{\lambda\sigma})\,\varepsilon^{\mu\nu\lambda\sigma}$ that cause the strong CP violation end up having coefficients that are required to vanish as a gauge choice in the process of reducing from flat 4+2 to flat 3+1 dimensions. This leads to the resolution of the strong CP problem [17].

Given these attractive features of the 4+2 approach, we are then tempted to propose a new gauge principle in interacting field theory in 3+1 dimensions:

> The 2T gauge principle - requiring that a theory in 3+1 dimensions
> be the gauge-fixed version of a 2T theory in 4+2 dimensions.

Remarkably, the Standard Model of Particles and Forces (SM) satisfies this principle [17] as already mentioned. While this was demonstrated by using a specific gauge (namely

the "massless particle gauge" in Table 1), the possibility of choosing other gauges listed in Tables 1,2, implies the existence of dual versions of the Standard Model in 3+1 dimensions.

Besides the resolution of the strong CP problem, and offering new points of views on the origin of mass, the 2T gauge principle enunciated above has further phenomenogical implications for unified theories, SUSY structures, and cosmology which still need to be worked out (for some comments see [17]). The duals of the Standard Model or its grand unified and/or supersymmetric extensions, are likely to suggest additional effects of phenomenological interest at the LHC.

The current paper is a first step in the investigation of 2T-physics dualities directly in the field theory formalism. For simplicity, in this paper, we will deal only with the 2T scalar field theory in $d+2$ dimensions. We will show that a particular class of gauge choices results in a family of Klein-Gordon (KG) action functionals for the conformal scalar field propagating on different spacetimes in $(d-1)+1$ dimensions and interacting locally.

The emergent spacetime metrics in our simplest class of examples use a generalization of the "relativistic massless particle" gauge to a family of metrics including the following special cases (even more interesting cases not in this class appear in Tables 1,2 below)

$$\begin{array}{l}\text{the flat spacetime,}\\ \text{the AdS}_{d-n}\times\text{S}^{n}\text{ spacetimes,}\\ \text{the maximally symmetric spacetimes,}\\ \text{the spacetime with a general function }\alpha\left(x\right),\\ \text{the Robertson-Walker cosmological spacetimes,}\\ \text{the cosmological constant spacetime,}\\ \text{the S}^{d-1}\times\text{R spacetime,}\\ \text{the general conformally flat spacetime}\end{array} \tag{2.1}$$

It will be shown that interacting KG field theory actions in these curved spaces have the following properties:

- Two such emergent field theories, with different background metrics $g_{\mu\nu}\left(x\right)$ and $\tilde{g}_{\mu\nu}\left(x\right)$ which are regarded in 1T-physics as different spacetimes, are related to each other by duality transformations. This is a consequence of the fact that all the emergent theories are gauge fixed versions of the same theory in $d+2$ dimensions. Remarkably,

this duality implies that the theories with non-trivial spacetimes listed above are all dual to the flat theory with the Minkowski metric $\eta_{\mu\nu}$.

- For each fixed background metric $g_{\mu\nu}(x)$ in $(d-1)+1$ dimensions listed above, the KG field action has a hidden SO$(d,2)$ global symmetry which is the same as the original SO$(d,2)$ symmetry of the 2T field theory in $d+2$ dimensions. The explicit form of the SO$(d,2)$ generators, expressed as transformations of the KG field $\phi(x)$ in the non-trivial backgrounds $g_{\mu\nu}(x)$ in $(d-1)+1$ dimensions, will also be given.

In the rest of the paper, we will first provide in section III a compendium of gauge choices in the context of the worldline theory. After that, in Section IV we first review the simplest gauge choice (massless particle gauge) in the context of field theory for a scalar field. This is the gauge that leads to the emergent Standard Model in flat $3+1$ spacetime as discussed in [17]. Then in Section VI, we will discuss a class of gauge choices that lead to conformally flat spacetimes in the context of field theory. Our general treatment is specialized to some interesting cases of emergent spacetimes listed in (2.1) that are often discussed in the literature in a variety of field theoretic applications. Our approach shows that the hidden SO$(d,2)$ symmetries in these field theories and the duality relations among them have often not been noticed in the past.

The hidden symmetries and dualities apply also to the case of the Standard Model, as will be shown in a companion paper. . They are verifiable directly in $(d-1)+1$ dimensions through computation and experiment. These, together with similar dualities that follow from more interesting gauge choices listed in Tables 1,2 (and commented on below), are just the tip of an "iceberg" signaling a unified master theory in $d+2$ dimensions.

III. GAUGE CHOICES

In this section, we provide a list of known gauge choices in the worldline formalism for the simplest 2T-physics systems, which is the free spinless particle in $d+2$ dimensions subject to the Sp$(2,R)$ gauge symmetry. These gauge choices have their equivalents in the field theory formalism, so it is useful to be guided by the worldline theory to understand their physical meaning in terms of 1T-physics.

The free 2T-physics particle in flat space is described by the action

$$S=\int d\tau \frac{1}{2}\varepsilon^{ij}\left(D_\tau X_i^M\right)X_j^N\eta_{MN},$$

where $X_i^M(\tau)\equiv\binom{X^M(\tau)}{P_M(\tau)}$, $i=1,2,$ is phase space considered as doublets under $\mathrm{Sp}(2,R)$, $\varepsilon^{ij}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}^{ij}$ is the antisymmetric $\mathrm{Sp}(2,R)$ metric, and $D_\tau X_i^M=\partial_\tau X_i^M-A_i{}^jX_j^M$ is the $\mathrm{Sp}(2,R)$ gauge covariant derivative, with the 3 gauge potentials $A^{ij}=\varepsilon^{ik}A_k{}^j=\begin{pmatrix}A&C\\C&B\end{pmatrix}$. More explicitly, after dropping a total derivative, we can write the action in the form

$$S=\int d\tau\left\{\dot{X}^MP_M-\frac{1}{2}AX\cdot X-\frac{1}{2}BP\cdot P-CX\cdot P\right\}. \tag{3.1}$$

The $\mathrm{Sp}(2,R)$ gauge generators are[2] $Q_{ij}=X_i\cdot X_j=(X\cdot X),(P\cdot P),(X\cdot P)$. This action has an evident global symmetry $\mathrm{SO}(d,2)$ as the symmetry of the flat metric η_{MN} in the dot product which defines the three $\mathrm{Sp}(2,R)$ gauge generators $X\cdot P=X^MP^N\eta_{MN}$, etc.. In the gauge fixed versions below the $\mathrm{SO}(d,2)$ symmetry will turn into a hidden global symmetry.

After two $\mathrm{Sp}(2,R)$ gauge choices, as given in the tables below, all components of X^M, P^M are expressed in terms of the remaining phase space $t,\vec{x},H,\vec{p}$ in $(d-1)+1$ dimensions. Then the action reduces to the form

$$S=\int d\tau\left\{\dot{\vec{x}}\cdot\vec{p}-\dot{t}H-\frac{1}{2}Bf\left(t,\vec{x},H,\vec{p}\right)\right\} \tag{3.2}$$

which describes a particle (not necessarily Lorentz or even rotation covariant) subject to a generalized τ reparametrization symmetry. In a subset of cases the action may take a more covariant form, such as $S=\int d\tau\left\{\dot{x}^\mu p_\mu-\frac{1}{2}Bg^{\mu\nu}(x)\,p_\mu p_\nu\right\}$. In the gauge $t(\tau)=\tau$, many Hamiltonians $H(\vec{x},\vec{p})$ emerge by solving the remaining constraint $f(t,\vec{x},H,\vec{p})=0$.

In the completely gauge fixed version with $\dot{t}=1$ the action becomes simply

$$S=\int d\tau\left\{\dot{\vec{x}}\cdot\vec{p}-H\left(\vec{x},\vec{p}\right)\right\} \tag{3.3}$$

in terms of the unconstrained phase space in 1T-physics. Different Hamiltonians emerge because time t has been embedded in the 2T phase space in 4+2 dimensions in different ways, as seen in the explicit parameterizations $X^M(t,\vec{x},H,\vec{p})$, $P^M(t,\vec{x},H,\vec{p})$ given in Tables 1,2 below. These give a compilation of some gauge choices that already appeared in previous

[2] More generally, the generators of $\mathrm{Sp}(2,R)$ are more complicated expressions $Q_{ij}(X,P)$ with $i,j=1,2$, that depend on the background fields. All possible background fields can be included [7].

papers, as well as other gauge choices that had remained unpublished. The tables are not exhaustive since all possible gauge choices are not known.

The two tables differ only in the choice of convenient components $M = (+', -', (m \text{ or } \mu))$ versus $M = (0', 0, 1', i)$ for parameterizing the gauge choices, where $X^{\pm'} = \frac{1}{\sqrt{2}}\left(X^{0'} \pm X^{1'}\right)$. The last column in Table 1 is labeled by μ in the simple case $X^\mu(x) = x^\mu$ and is labeled by m otherwise, where m implies $(\mu \oplus i)$ or more general possibilities. The total number of dimensions labelled by $m = \mu$ or $m = (\mu \oplus i)$ or $I = (1', i)$ is always d.

Explanatory comments on the entries in both tables will be given following general remarks. For more detailed information on these gauges the reader can consult [3]-[5].

Gauge choice		$+'$	$-'$	$m = (\mu \oplus i),\ \mu = 0, 1, \cdots$
Relativistic massless particle	$X^M =$	1	$\frac{1}{2}x^2$	x^μ
$p^2 = 0$	$P^M =$	0	$x \cdot p$	p^μ
$\mathrm{AdS}_{d-n} \times \mathrm{S}^n$	$X^M =$	$\frac{R_0^2}{\lvert\vec{y}\rvert}$	$\frac{1}{2\lvert\vec{y}\rvert}(x^2 + \vec{y}^2)$	$\frac{R_0}{\lvert\vec{y}\rvert}x^\mu,\ \frac{R_0}{\lvert\vec{y}\rvert}\vec{y}^i$
$\vec{y}^2(p^2 + \vec{k}^2) = 0$	$P^M =$	0	$\frac{\lvert\vec{y}\rvert}{R_0^2}(x \cdot p + \vec{y} \cdot \vec{k})$	$\frac{\lvert\vec{y}\rvert}{R_0}p^\mu,\ \frac{\lvert\vec{y}\rvert}{R_0}\vec{k}^i$
Maximally Symmetric Spaces	$X^M =$	$1+\sqrt{1-Kx^2}$	$\frac{x^2/2}{1+\sqrt{1-Kx^2}}$	x^μ
$p^2 - \frac{K\ (x \cdot p)^2}{1-Kx^2} = 0$	$P^M =$	0	$\frac{\sqrt{1-Kx^2}}{1+\sqrt{1-Kx^2}}x \cdot p$	$p^\mu - \frac{Kx \cdot p\ x^\mu}{1+\sqrt{1-Kx^2}}$
Free function $\alpha(x)$	$X^M =$	$x^2 + \alpha(x)$	$\frac{x^2/2}{x^2+\alpha(x)}$	x^μ
$p^2 + \frac{4\alpha(x)(x \cdot p)^2}{(x^2-\alpha(x))^2} = 0$	$P^M =$	0	$\frac{x \cdot p}{\alpha(x)-x^2}$	$p^\mu - \frac{2x \cdot p}{x^2-\alpha(x)}x^\mu$
Conformally flat $g_{\mu\nu}=e_\mu^m(x)e_\nu^n(x)\eta_{mn}$	$X^M =$	$\pm e^{\sigma(x)}$	$\pm\frac{1}{2}e^{\sigma(x)}q^2(x)$	$\pm e^{\sigma(x)}q^m(x^\mu)$ $e_\mu^m(x) \equiv \pm e^{\sigma(x)}\frac{\partial q^m(x)}{\partial x^\mu}$
$g^{\mu\nu}(x)p_\mu p_\nu = 0$	$P^M =$	0	$q^m(x)e_m^\mu(x)p_\mu$	$e_m^\mu(x)p_\mu$
Relativistic massive particle	$X^M =$	$\frac{1+a}{2a}$	$\frac{x^2 a}{1+a}$	x^μ $\quad a \equiv \sqrt{1+\frac{m^2x^2}{(x \cdot p)^2}}$
$p^2 + m^2 = 0$	$P^M =$	$\frac{-m^2}{2ax \cdot p}$	$a\ x \cdot p$	p^μ
Non-relativistic massive particle	$X^M =$	t	$\frac{\mathbf{r}\cdot\mathbf{p}-tH}{m}$	$X^0 = \pm\left\lvert\mathbf{r}-\frac{t}{m}\mathbf{p}\right\rvert,\ X^i = \mathbf{r}^i$
$H - \frac{\mathbf{p}^2}{2m} = 0$	$P^M =$	m	H	$P^0 = 0,\ P^i = \mathbf{p}^i$

Table1: Parametrization of X^M, P^M for $M = (+', -', (m \text{ or } \mu))$

In each gauge choice two degrees of freedom in $\left(X^M(\tau), P_M(\tau)\right)$ have been gauge fixed for all τ and the two constraints $X^2 = X \cdot P = 0$ have been explicitly solved to give all components of X^M, P_M in terms of the remaining degrees of freedom $t, \vec{x}, H, \vec{p}$. The third $\mathrm{Sp}(2,R)$ constraint $P^2 = 0$ is equivalent to the constraint among the remaining degrees of

freedom $t, \vec{x}, H, \vec{p}$ as listed in the first column.

For example, in the case of the massive non-relativistic particle in Table 1, the two gauge choices are $P^{+'}(\tau) = m$ and $P^0(\tau) = 0$, while the solution of $X^2 = X \cdot P = 0$ is given by $X^{-'} = \frac{\vec{r}\cdot\vec{p}-tH}{m}$ and $X^0 = \pm|\vec{r}-t\vec{p}/m|$, where $t(\tau)$ is a function of τ and is canonical to $H(\tau)$. For the remaining gauge symmetry we can choose $t(\tau) = \tau$ and the remaining constraint gives $0 = P^2 = -2P^{+'}P^{-'} - P_0^2 + P_i^2 = -2mH + 0 + \vec{p}^2$, which is solved by the non-relativistic Hamiltonian $H = \frac{\vec{p}^2}{2m}$ listed in the first column.

Gauge choice	M	$0'$	0	$I = (1', i)$
Robertson-Walker $r<R_0$ (closed universe)	$X^M =$	$a(t)\cos(\int^t \frac{dt'}{a(t')})$	$a(t)\sin(\int^t \frac{dt'}{a(t')})$	$X^i{=}\mathbf{r}^i a(t)/R_0$ $X^{1'}{=}\pm a(t)\sqrt{1-\frac{r^2}{R_0^2}}$
$-H^2+\frac{R_0^2}{a^2(t)}(\mathbf{p}^2-\frac{(\mathbf{r}\cdot\mathbf{p})^2}{R_0^2})=0$	$P^M =$	$-H\sin(\int^t \frac{dt'}{a(t')})$	$H\cos(\int^t \frac{dt'}{a(t')})$	$P^i{=}\frac{R_0}{a(t)}(\mathbf{p}^i-\frac{\mathbf{r}\cdot\mathbf{p}}{R_0^2}\mathbf{r}^i)$ $P^{1'}{=}\mp\frac{\mathbf{r}\cdot\mathbf{p}}{a(t)}\sqrt{1-\frac{r^2}{R_0^2}}$
Robertson-Walker $r>0$ (open universe)	$X^M =$	$a(t)\sinh(\int^t \frac{dt'}{a(t')})$	$(\pm)' a(t)\sqrt{1+\frac{r^2}{R_0^2}}$	$X^i{=}\mathbf{r}^i a(t)/R_0$ $X^{1'}{=}\pm a(t)\cosh(\int^t \frac{dt'}{a(t')})$
$-H^2+\frac{R_0^2}{a^2(t)}(\mathbf{p}^2+\frac{(\mathbf{r}\cdot\mathbf{p})^2}{R_0^2})=0$	$P^M =$	$\pm H\cosh(\int^t \frac{dt'}{a(t')})$	$(\pm)'\frac{\mathbf{r}\cdot\mathbf{p}}{a(t)}\sqrt{1+\frac{r^2}{R_0^2}}$	$P^i{=}\frac{R_0}{a(t)}(\mathbf{p}^i+\frac{\mathbf{r}\cdot\mathbf{p}}{R_0^2}\mathbf{r}^i)$ $P^{1'}{=}H\sinh(\int^t \frac{dt'}{a(t')})$
Cosmological constant $\Lambda\equiv \frac{3}{R_0^2} > 0$	$X^M =$	$\sqrt{R_0^2-r^2}\sinh\frac{t}{R_0}$	R_0	$X^i{=}\mathbf{r}^i$ $X^{1'}{=}\pm\sqrt{R_0^2-r^2}\cosh\frac{t}{R_0}$
$-H^2(1-\frac{r^2}{R_0^2})+(\mathbf{p}^2+\frac{(\mathbf{r}\cdot\mathbf{p})^2}{R_0^2-r^2})=0$	$P^M =$	$\pm\frac{H}{R_0}\sqrt{R_0^2-r^2}\cosh\frac{t}{R_0}$	$\frac{R_0\mathbf{r}\cdot\mathbf{p}}{R_0^2-r^2}$	$P^i{=}\mathbf{p}^i+\frac{\mathbf{r}\cdot\mathbf{p}}{R_0^2-r^2}\mathbf{r}^i$, $P^{1'}{=}\frac{H}{R_0}\sqrt{R_0^2-r^2}\sinh\frac{t}{R_0}$
Cosmological constant $\Lambda\equiv -\frac{3}{R_0^2} < 0$	$X^M =$	$\sqrt{R_0^2+r^2}\sin\frac{t}{R_0}$	$\mp\sqrt{R_0^2+r^2}\cos\frac{t}{R_0}$	$X^i{=}\vec{r}^{\,i}$ $X^{1'}{=}R_0$
$-H^2(1+\frac{r^2}{R_0^2})+(\mathbf{p}^2-\frac{(\mathbf{r}\cdot\mathbf{p})^2}{R_0^2+r^2})=0$	$P^M =$	$\pm\frac{H}{R_0}\sqrt{R_0^2+r^2}\cos\frac{t}{R_0}$	$\frac{H}{R_0}\sqrt{R_0^2+r^2}\sin\frac{t}{R_0}$	$P^i{=}\mathbf{p}^i-\frac{\mathbf{r}\cdot\mathbf{p}}{R_0^2+r^2}\mathbf{r}^i$, $P^{1'}{=}-\frac{R_0\mathbf{r}\cdot\mathbf{p}}{R_0^2+r^2}$
(d–1)-sphere$\times$time	$X^M =$	$R_0\cos\frac{t}{R_0}$	$R_0\sin\frac{t}{R_0}$	$R_0\hat{n}^I =$ $X^i{=}\mathbf{r}^i$ $X^{1'}{=}\pm\sqrt{R_0^2-r^2}$
$-H^2+(\mathbf{p}^2+\frac{(\mathbf{r}\cdot\mathbf{p})^2}{R_0^2-r^2})=0$	$P^M =$	$-H\sin\frac{t}{R_0}$	$H\cos\frac{t}{R_0}$	$P^i{=}\mathbf{p}^i$ $P^{1'}{=}\mp\frac{\mathbf{r}\cdot\mathbf{p}}{\sqrt{R_0^2-r^2}}$
H-atom, $H<0$	$X^M =$	$r\cos u$ $u(t)\equiv\frac{\sqrt{-2mH}}{m\alpha}(\mathbf{r}\cdot\mathbf{p}-2mHt)$	$r\sin u$	$X^i{=}\mathbf{r}^i-\frac{r}{m\alpha}\mathbf{r}\cdot\mathbf{p}\ \mathbf{p}^i$ $X^{1'}{=}-\frac{r}{m\alpha}\sqrt{-2mH}\mathbf{r}\cdot\mathbf{p}$
$H=\frac{\mathbf{p}^2}{2m}-\frac{\alpha}{r}$	$P^M =$	$-\frac{m\alpha}{r\sqrt{-2mH}}\sin u$	$\frac{m\alpha}{r\sqrt{-2mH}}\cos u$	$P^i{=}\mathbf{p}^i$ $P^{1'}{=}\frac{1}{\sqrt{-2mH}}\left(\frac{m\alpha}{r}-\mathbf{p}^2\right)$
H-atom, $H>0$	$X^M =$	$r\cosh u$ $u(t)\equiv\frac{\sqrt{2mH}}{m\alpha}(\mathbf{r}\cdot\mathbf{p}-2mHt)$	$\frac{r}{m\alpha}\sqrt{2mH}\mathbf{r}\cdot\mathbf{p}$	$X^i{=}\mathbf{r}^i-\frac{r}{m\alpha}\mathbf{r}\cdot\mathbf{p}\ \mathbf{p}^i$ $X^{1'}{=}r\sinh u$
	$P^M =$	$\frac{m\alpha}{r\sqrt{2mH}}\sinh u$	$\frac{1}{\sqrt{2mH}}\left(\frac{m\alpha}{r}-\mathbf{p}^2\right)$	$P^i{=}\mathbf{p}^i$ $P^{1'}{=}\frac{m\alpha}{r\sqrt{2mH}}\cosh u$

Table2 : Parametrization of X^M, P^M for $M = (0', 0, I)$

The different Hamiltonians $H(\vec{x}, \vec{p})$ that emerge from such gauge choices are all holographic representatives of the same master theory in Eq.(3.1) and therefore must form a

set of dual theories that are transformed into each other by the original gauge symmetry $\mathrm{Sp}(2,R)$. Furthermore the emergent actions all must have hidden global symmetry $\mathrm{SO}(d,2)$. The generators of $\mathrm{SO}(d,2)$ are $L^{MN}=\varepsilon^{ij}X_i^M X_j^N = X^M P^N - X^M P^N$, which are evidently gauge invariant under $\mathrm{Sp}(2,R)$, and are conserved $\partial_\tau L^{MN}=0$ by using the original equations of motion that follow from (3.1). In the respective phase spaces these take the following non-linear forms

$$L^{MN}=X^M\left(t,\vec{x},H,\vec{p}\right)\ P^N\left(t,\vec{x},H,\vec{p}\right)-X^N\left(t,\vec{x},H,\vec{p}\right)\ P^M\left(t,\vec{x},H,\vec{p}\right). \tag{3.4}$$

Under Poisson brackets in the respective phase spaces these L^{MN} close to form the Lie algebra of $\mathrm{SO}(d,2)$. Here (t,H) can be treated as canonical conjugates. But, it is also possible to make the final gauge choice $t(\tau)=\tau$; in that case τ is treated as a constant and H is replaced by the solution of the last constraint $f\left(t,\vec{x},H,\vec{p}\right)=0$, which gives $H=H\left(\vec{x},\vec{p}\right)$ as dependent on the remaining canonical variables.

In the quantum theory, canonical conjugates must be ordered to insure that the L^{MN} satisfy the $\mathrm{SO}(d,2)$ Lie algebra. This was done successfuly for some of the examples in Tables 1,2 in the first quantization approach [3][4]. In the field theory context discussed in this paper, the quantum ordering is automatically achieved, as discussed in section VIII.

The L^{MN} are constants of motion and they have the same gauge invariant value in each phase space. So any function of the L^{MN} has identical values in any of the phase spaces that appear in Tables 1,2. Therefore the L^{MN} are the key to the dualities among these systems. The duality transformation is the original $\mathrm{Sp}(2,R)$, which transforms one phase space (a given gauge choice) to another. Through the gauge invariant (or duality invariant) L^{MN} one can establish an infinite set of duality relations among these systems. These can be checked through computation or through experiment. In the first quantized theory, one consequence of this duality is that all of the systems in Tables 1,2 provide Hilbert spaces that must span the same representation of $\mathrm{SO}(d,2)$ (but in different bases defined by diagonalizing the Hamiltonian H). The $\mathrm{SO}(d,2)$ Casimir eigenvalues C_n of this universal representation are independent of the gauge choice; for example the quadratic Casimir eigenvalue at the quantum level is given by $C_2=\frac{1}{2}L^{MN}L_{MN}=1-d^2/4$ [1]. The universal value of the Casimirs is one of the consequences of duality that is independent of the details of a particular quantum state, and can be verified easily to be true for each physical system described in the Tables.

Similarly all the other Casimirs operators are fixed[3], and the resulting unitary representation is the singleton of $\mathrm{SO}(d,2)$. In the same singleton space, each system in Tables 1,2 corresponds to a different choice of basis labelled by the eigenvalues of simultaneous observables, one of which is the choice of Hamiltonian H (choice of time) in the respective phase space. The transformation $\langle basis\ 1|\ basis\ 2\rangle$ from one complete basis to another within the same singleton representation should be understood as the duality transformation at the quantum level.

The following comments give further information on the entries in Table 1.

1. For the maximally symmetric space the canonical pairs are (x^{μ}, p_{μ}). The curved space on which the particle propagates is determined by the flat $\mathrm{SO}(d,2)$ metric $ds^2 = dX^M dX^N \eta_{MN} = -2dX^{+'}dX^{-'} + dX^{\mu}dX^{\nu}\eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric in $(d-1)+1$ dimensions. This metric becomes $ds^2 = dx^{\mu}dx^{\nu}g_{\mu\nu}(x)$, with $g_{\mu\nu} = \eta_{\mu\nu} + \frac{K}{1-Kx^2}x_{\mu}x_{\nu}$. The constraint listed in the first column $P\cdot P = p^2 - \frac{K\ (x\cdot p)^2}{1-Kx^2} = g^{\mu\nu}(x)\,p_{\mu}p_{\nu} = 0$ involves the inverse of the metric. When the canonical operators are properly ordered in the quantum theory, this constraint becomes the Laplacian in curved space for the conformal scalar (including the curvature term) given in Eq.(3.8), as we will show[4] in section VI. The Riemann scalar curvature for this space is a constant $R = K$, and evidently it reduces to flat space (first entry in Table 1) if the curvature modulus K in the gauge choice vanishes $K \to 0$. Some well known maximally symmetric spaces include DeSitter space with $K > 0$ and Anti-deSitter space with $K < 0$. For more information on maximally symmetric spaces see [21].

2. For $\mathrm{AdS}_{d-n}\times \mathrm{S}^n$, the canonical pairs are the $(x^{\mu}(\tau), p_{\mu}(\tau))$ in $d-n-1$ dimensions and the $(y^i(\tau), k_i(\tau))$ in $n+1$ dimensions, for a total of d dimensions. The curved space on which the particle propagates is given by $ds^2 = dX^M dX^N \eta_{MN} = \frac{R_0^2}{y^2}\left[(dx^{\mu})^2 + (d\overrightarrow{y}^i)^2\right] = \frac{R_0^2}{y^2}\left((dx)^2 + (dy)^2\right) + R_0^2\,(d\Omega_n)^2$. In the last expression $y \equiv |\vec{y}|$

[3] In the classical theory $C_2 = 0$, and similarly all $C_n = 0$, follow from the constraints $X^2 = P^2 = X\cdot P = 0$. But in the quantum theory, ordering of operators lead to the non-zero eigenvalues of C_n that correspond to the singleton representation.

[4] The same result would follow in the first quantization approach by ordering properly the $L^{MN}(t, \vec{x}, H, \vec{p})$ and insuring that the quantum constraint listed in the first column is invariant (or properly transforms) under it. This was the method used in [4] as demonstrated for the $\mathrm{AdS}_{d-n}\times \mathrm{S}^n$ case. In the present paper, the field theory approach automatically takes care of all ordering issues.

and then $\frac{R_0^2}{y^2}\left((dx)^2+(dy)^2\right)$ is the AdS$_{d-n}$ metric, while $(d\Omega_n)^2$ is the S^n metric build from the unit vector $\vec{y}/y$ embedded $n+1$ dimensions. Note that our construction in terms of the SO$(d,2)$ vector X^M shows that the full symmetry of the spacetime AdS$_{d-n}\times$S^n is SO$(d,2)$ and not only SO$(d-n-1,2)\times$SO$(n+1)$ as it is it often mentioned in the literature (see [4] for more details).

3. The metric that emerges in the case of the free function $\alpha(x)$ is of the form: $g_{\mu\nu}=\eta_{\mu\nu}-\frac{4\alpha(x)}{(x^2+\alpha(x))^2}x_\mu x_\nu$, where $\alpha(x)$ is allowed to be any function of x^μ. If we specialize to the case of a constant α we see that the space is asymptotically flat and we obtain simple expressions for its curvature tensors

$$R_{\lambda\sigma\mu\nu}=\frac{-4\alpha}{(x^2-\alpha)^2}\eta_{\lambda[\mu}\eta_{\nu]\sigma}-\frac{8\alpha}{(x^2-\alpha)^2(x^2+\alpha)}x_{[\mu}\eta_{\nu][\lambda}x_{\sigma]}\quad(\text{if }\alpha=\text{constant })\tag{3.5}$$

$$R_{\lambda\mu}=\eta_{\lambda\mu}\left(4\alpha\frac{(d-1)\alpha-x^2(d-3)}{(x^2-\alpha)^3}\right)+x_\lambda x_\mu\left(8\alpha\frac{x^2(d-2)-d\alpha}{(x^2-\alpha)^3(x^2+\alpha)}\right)\tag{3.6}$$

$$R=\frac{-4\alpha}{(x^2-\alpha)^2}\left(d^2+d\frac{x^2+\alpha}{x^2-\alpha}-\frac{2x^2(x^2+\alpha)^2}{(x^2-\alpha)^3}\right)\tag{3.7}$$

There are curvature singularities at $x^2=\alpha$, and there are also values of x^2 where the curvature scalar vanishes. If $\alpha(x)$ is allowed to be a function rather than a constant then the expressions for the curvature tensors are more involved, but generically we expect a space with curvature singularities and zeroes.

4. The conformally flat case is the most general gauge in which all components of position $X^M(x)$ in $d+2$ dimensions are a functions of only position x^μ (and not momentum p_μ) in $(d-1)+1$ dimensions. The main part of this paper starting with the next section will be involved with the equivalent of this gauge choice in the context of 2T field theory. The conformally flat case is a generalized version of the four items that precedes it in Table-1. $q^m(x)$ and $\sigma(x)$ can be chosen arbitrarily as functions of the spacetime coordinates $x^\mu(\tau)$, while the canonical conjugates are $(x^\mu(\tau),p_\mu(\tau))$. The curved space on which the particle propagates is given by $ds^2=dX^MdX^N\eta_{MN}=dx^\mu dx^\nu g_{\mu\nu}(x)$ with

$$g_{\mu\nu}(x)=e_\mu^m(x)\,e_\nu^n(x)\,\eta_{mn},\quad e_\mu^m(x)=e^{\sigma(x)}\frac{\partial q^m(x)}{\partial x^\mu},\tag{3.8}$$

where η_{mn} is the flat Minkowski metric in d dimensions. This is the general conformally flat metric. Evidently this general parametrization reproduces the maximally

symmetric space by taking $e^{\sigma(x)}q^m(x) = \delta^m_\mu x^\mu$ and $e^{\sigma(x)} = 1+\sqrt{1-Kx^2}$. It also reproduces the $\mathrm{AdS}_{d-n}\times\mathrm{S}^n$ and the free-function-α cases by taking $q^m(x) = \frac{1}{R_0}\left(x^\mu, \overrightarrow{y}^i\right)$ with $e^{\sigma(x)} = \frac{R_0^2}{|\overrightarrow{y}|}$ for $\mathrm{AdS}_{d-n}\times\mathrm{S}^n$, and $e^{\sigma(x)}q^m(x) = \delta^m_\mu x^\mu$ with $e^{\sigma(x)} = \frac{x^2}{x^2+\alpha(x)}$ for free-function-α. The curvature tensors of the general conformally flat space (3.8) are computed in the Appendix. In particular the scalar curvature is given by $R = d(1-d)\left[g^{\mu\nu}\left(\partial_\mu\sigma\partial_\nu\sigma + \frac{2}{d}\partial_\mu\partial_\nu\sigma\right) + \frac{2}{d}e^{\nu m}\left(\partial_\nu e^\mu_m\right)\partial_\mu\sigma\right]$ as in Eq.(A7).

5. The massive relativistic and non-relativistic gauges are distinguished from the others in Table-1 by the fact that the expressions for some of the positions $X^M(x,p)$ involve momenta p. This does not happen with the other gauges in which the positions $X^M(x)$ are functions of only positions x. In the latter case $X^M(x)$ we always get a particle moving on some curved space as explained above. However, when positions and momenta are mixed in the gauge choices $X^M(x,p)$, the emerging dynamics is more intricate and cannot be described as due to curved space only. We will make more comments on this feature at the end of this section.

The following comments give further information on the entries in Table 2. As in Table 1, the emergent spacetimes in $(d-1)+1$ dimensions for the cases $X^M(x)$ are all described by $ds^2 = dX^M dX^N \eta_{MN} = -\left(dX^{0'}\right)^2 - \left(dX^0\right)^2 + \left(dX^{1'}\right)^2 + \left(d\vec{X}\right)^2 = g_{\mu\nu}(x)\,dx^\mu dx^\mu$, which shows that they all are symmetric under the hidden $\mathrm{SO}(d,2)$ global symmetry.

1. In the Robertson-Walker case the metric is given by

$$ds^2 = -dt^2 + a^2(t)\left[\left(1 \mp \frac{r^2}{R_0^2}\right)^{-1} dr^2 + r^2\left(d\Omega_{d-2}\right)^2\right] \begin{array}{l}(-)\ \ r < R_0 \\ (+)\ \ r > 0\end{array} \tag{3.9}$$

In a cosmological context this, and its more specialized Friedman universe version, is the spacetime that describes the evolution of the universe as a whole. The $(-)$ and $(+)$ cases correspond to closed and open universes respectively.

2. The gauge labelled as the cosmological constant describes a particle moving in free space except for the influence of a cosmological constant Λ. The metric in this case is given by

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2\left(d\Omega_{d-2}\right)^2 \tag{3.10}$$

for either positive or negative $\Lambda = \pm\frac{3}{R_0^2}$. This a particular form of the deSitter $(\Lambda > 0)$ or anti-deSitter $(\Lambda < 0)$ metric.

3. In the case of the $(d-1)$ sphere$\times$time, the particle moves on a spacetime described by the metric
$$ds^2 = dX^M dX^N \eta_{MN} = -dt^2 + R_0^2 \left(d\Omega_{d-1}\right)^2, \tag{3.11}$$
The metric on the sphere $R_0^2 \left(d\Omega_{d-1}\right)^2$ is built from the unit vector n^I in d dimensions. If $n^I\left(\vec{r}\right)$ is parameterized as given in the table, the metric takes the form $ds^2 = -dt^2 + \left(1-\frac{r^2}{R_0^2}\right)^{-1} dr^2 + r^2 \left(d\Omega_{d-2}\right)^2$ for $r < R_0$.

4. The H-atom gauge is fairly complex since it involves the type of parametrization $X^M\left(x,p\right)$ which includes both position and momentum[5]. It is worth noting that the vector $X^I/r = (X^{1'}, \vec{X})/r$ as parameterized on the table in the case of $H<0$ is a unit vector $\left(X^I/r\right)^2 = 1$ embedded in d dimensions. Similarly the momentum $rP^I = r(P^{1'}, \vec{P})$ is also a unit vector. For $d=4$, this explains the SO(4) symmetry of the H-atom (or planetary system) Hamiltonian as being due to rotation symmetry in four space dimensions. Including all the $d+2$ coordinates, one learns that the non-relativistic *action* that describes the H-atom (i.e. particle in $1/r$ potential) has the hidden symmetry SO$(d,2)$, as do all the other sytems listed in Tables 1,2.

5. Additional gauge choices of the type $X^M\left(x,p\right)$ which includes both position and momentum are easy to generate from the ones listed above by interchanging the roles of position/momentum for some of the entries in the process of choosing gauges.

Our focus in this paper is field theory. The first goal is to find the analogs of gauge choices displayed above in the language of field theory, and use them to derive many 1T-physics field theories from the same 2T-physics field theory. This will establish a set of dual field theories which could be used as a technical tool for computations, as well as for the purposes of unification leading to a deeper understanding of Nature.

We will see that for the gauge fixing of the 2T field theory we will also need the curvature scalar for the metrics that appeared in Tables 1-2. Therefore we collect here the Ricci scalar for these metrics. The Ricci scalar for the conformally flat scalar in the last item is computed

[5] The $\left(X^M, P^M\right)$ in our table is related to the $\left(\tilde{X}^M, \tilde{P}^M\right)$ for the H-atom given in a previous publication [3], by the relation $X^M = r\tilde{X}^M$ and $P^M = \frac{1}{r}\tilde{P}^M$. This is an Sp$(2,R)$ transformation that does not change the meaning of time or Hamiltonian. Furthermore, compared to [3] we have replaced α by $m\alpha$.

in the Appendix.

	metric	curvature scalar R
Flat space	$ds^2 = dx \cdot dx \equiv \eta_{\mu\nu} dx^\mu dx^\nu$	0
AdS$_{d-n}\times$S^n	$ds^2 = \frac{R_0^2}{y^2}(dx \cdot dx + dy^2) + R_0^2 (d\Omega_n)^2$	$\frac{1}{R_0^2}(2n-d)(d-1)$
Maximally symm.	$ds^2 = dx \cdot dx + \frac{K}{1-Kx^2}(x \cdot dx)^2$	K
Robertson-Walker $a(t)$	$ds^2 = -dt^2 + a^2 \left\{ \begin{array}{c} \frac{1}{1\mp r^2/R_0^2} dr^2 \\ +r^2 (d\Omega_{d-2})^2 \end{array} \right\}$	$(d-2)(d-1) \left\{ \begin{array}{c} \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 \\ \pm \frac{1}{a^2 R_0^2} \end{array} \right\}$
Cosmological const.	$ds^2 = \left\{ \begin{array}{c} -\left(1-\frac{\Lambda}{3}r^2\right) dt^2 \\ +\frac{dr^2}{1-\frac{\Lambda}{3}r^2} + r^2 (d\Omega_{d-2})^2 \end{array} \right\}$	$d\Lambda$
$(d-1)$-sphere$\times$time	$ds^2 = -dt^2 + R_0^2 (d\Omega_{d-1})^2$	$\frac{1}{R_0^2}(d-2)(d-1)$
Free function $\alpha(x)$	$ds^2 = dx \cdot dx + \frac{4\alpha(x)}{(x^2+\alpha(x))^2}(x \cdot dx)^2$	$\frac{-4\alpha}{(x^2-\alpha)^2}\left(d^2 + d\frac{x^2+\alpha}{x^2-\alpha} - \frac{2x^2(x^2+\alpha)^2}{(x^2-\alpha)^3}\right)$ for α=constant
Conformally flat	$ds^2 = e^{2\sigma(x)} \frac{q^m(x)}{\partial x^\mu} \frac{q_m(x)}{\partial x^\nu} dx^\mu dx^\nu$	$(1-d) \left\{ \begin{array}{c} d g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \\ +2 g^{\mu\nu} (\partial_\mu \partial_\nu \sigma) \\ +2 e^{\nu m} \partial_\nu e_m^\mu \partial_\mu \sigma \end{array} \right\}$

Table 3 - Curvature scalar for metrics in Tables 1,2. (3.12)

As we will show, all the cases in Tables 1,2 which do not involve momenta in the gauge choices of $X^M(x)$ are easily reproduced in 2T field theory. For this reason in this first paper we concentrate on these cases in 2T field theory as presented in Sections IVV and VI. The cases that mix position and momentum in the gauge choices for $X^M(x,p)$, such as the examples of the massive particles, including relativistic, non-relativistic or H-atom gauges, are harder because the $d+2$ field theory is formulated by making a distinction between position and momentum X^M and $P_M \to -i\partial_M$ at the outset (see [14] for some discussion of the non-relativistic case in field theory). Since these cases involve mass, which emerges as a modulus from the higher dimensions, they seem rather interesting as a notion that could relate to the origin of mass. We plan to discuss this issue in a future paper.

IV. D+2 FIELD THEORY

2T field theory has been fully formulated for fields of spins $0, \frac{1}{2}, 1$ [17], and to a lesser extent for spin 2 [14] and beyond [7], and has also been supersymmetrized [18]. The scalar

field provides a first example in the present paper for multiple gauge fixing. Its 2T action is

$$S(\Phi) = Z \int \left(d^{d+2}X\right) \delta\left(X^2\right) \left[\frac{1}{2}\Phi\partial^2\Phi - \gamma\frac{d-2}{2d}\Phi^{\frac{2d}{d-2}}\right], \tag{4.1}$$

Here Z is an overall normalization constant that will be determined.

The action was obtained through a BRST procedure [16] consistent with the underlying Sp$(2,R)$ gauge symmetry of the worldline theory (3.1). The equations of motion Eqs.(4.5) that follow from this action impose the Sp$(2,R)$ gauge singlet conditions, $X^2 = X\cdot P = P^2 = 0$, in the free field case. These free field equations are equivalent to covariant first quantization of the worldline theory. The BRST approach of [16] allows interactions of the special form $V(\Phi) = \gamma\frac{d-2}{2d}\Phi^{\frac{2d}{d-2}}$, which is the renormalizable Φ^4 interaction for $d+2 = 4+2$.

Note, in particular the presence of the delta function in the action, which imposes the $X^2 = 0$ condition, and which is crucial to obtaining three ($X^2 = 0$, kinematic, and dynamic) 2T equations of motion Eqs.(4.5) from this action with a single field [17]. It should also be emphasized that, due to this $\delta(X^2)$, the action is not invariant under translations of X^M. However, one should realize that, for some gauge choices, the translations in the lower dimensions are included in the original SO$(d,2)$ symmetry. Indeed, in the next section we will describe the emergent field theory in flat spacetime in the $(d-1)+1$ dimensions that is obtained by considering the massless particle gauge of Table 1. The SO$(d,2)$ symmetry will then be interpreted as conformal symmetry, which includes the Poincaré symmetry with generators $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ and $L^{+'\mu} = p^\mu$ as computed directly from Table 1.

The simplified version of the BRST gauge symmetry of [16] was called 2T-gauge-symmetry in [17], and is given by $\delta_\Lambda\Phi(X) = X^2\Lambda(X)$. In the simplified version it is useful to think of $\Phi(X)$ in the form $\Phi(X) = \Phi_0(X) + X^2\tilde{\Phi}(X)$, where $\Phi_0 \equiv [\Phi(X)]_{X^2=0}$ is defined as $\Phi(X)$ evaluated at $X^2 = 0$, and $X^2\tilde{\Phi}(X) \equiv \Phi(X) - \Phi_0(X)$ is the remainder. According to the gauge transformation, we see that the remainder $\tilde{\Phi}$ is gauge freedom[6]. It is sufficient to consider the simplified 2T-gauge-symmetry to uniquely determine the form of the action given above. This gauge symmetry automatically prevents the appearance of

[6] In this simplified form both $\Lambda(X)$ and $\tilde{\Phi}(X)$ are a priori taken as homogeneous fields that satisfy the homogeneity conditions $\left(X\cdot\partial + \frac{d+2}{2}\right)\tilde{\Phi}(X) = 0$, and similarly for $\Lambda(X)$. There is a more complete, but more elaborate form of the gauge transformation $\delta_\Lambda\Phi = \Lambda_0 + X^2\Lambda_1$, with a relation between Λ_0 and Λ_1, that leaves the same action invariant. With the stronger form of the gauge symmetry the homogeneity restriction is lifted so that $\tilde{\Phi}(X)$ is arbitrary. However, using the stronger form we can choose an intermediate gauge that makes $\tilde{\Phi}(X)$ homogeneous as above, or gauge fix it to zero directly.

any other power of the field in the potential $V(\Phi)$, including a quadratic mass term[7]. As a consequence, the $d+2$ theory cannot have a mass term and is invariant under global scale transformations $\Phi'(X) = e^{a(d-2)/2}\Phi(e^a X)$. This scale transformation in $d+2$ dimensions is separate from the manifest global symmetry $\mathrm{SO}(d,2)$. Note that $\mathrm{SO}(d,2)$ includes a transformation with generator $D \equiv L^{+'-'}$ which turns into a scale transformation in the lower dimension x^μ when $\mathrm{SO}(d,2)$ metamorphoses into conformal symmetry in the massless particle gauge of Table 1.

Varying the action gives the Euler-Lagrange equation of motion

$$\delta\left(X^2\right)\left[\partial^2\Phi - V'(\Phi)\right] + 2\delta'\left(X^2\right)\left(X\cdot\partial + \frac{d-2}{2}\right)\Phi = 0. \tag{4.2}$$

This results in two independent equations of motion as coefficients of $\delta(X^2)$ and $\delta'(X^2)$

$$\left[\left(X\cdot\partial + \frac{d-2}{2}\right)\Phi_0\right]_{X^2=0} = 0 \tag{4.3}$$

$$\left[\partial^2\Phi_0 - V'(\Phi_0) - 2\left(X\cdot\partial + \frac{d+2}{2}\right)\tilde{\Phi}\right]_{X^2=0} = 0, \tag{4.4}$$

where we take into account that the equation $\delta(X^2)F(X) + \delta'(X^2)G(X) = 0$ has the general solution $(G_0)_{X^2=0} = 0$ and $\left(F - \tilde{G}\right)_{X^2=0} = 0$, where $G(X) = G_0(X) + X^2\tilde{G}(X)$. Using the gauge symmetry we can choose the gauge $\tilde{\Phi} = 0$, or else impose the homogeneity condition described in footnote (6). Then the equations can be rewritten more simply as

$$\left[\partial^2\Phi - V'(\Phi)\right]_{X^2=0} = 0, \text{ and } \left[\left(X\cdot\partial + \frac{d-2}{2}\right)\Phi\right]_{X^2=0} = 0. \tag{4.5}$$

These, together with $X^2 = 0$, are the three 2T equations of motion for a scalar field that correspond to the three $\mathrm{Sp}(2,R)$ constraints, including interactions. They were originally found in [22][23][14] at the level of equations of motion[8], but now we can derive them from an action principle which is needed, among other things, for the field quantization of the theory.

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[7] However, if there are several fields, their products may appear as long as the total scale dimension is d, to be cancelled by the scale of the volume element $\left(d^{d+2}X\right)\delta\left(X^2\right)$.

[8] The $\mathrm{Sp}(2,R)$ point of view developed as an independent fundamental principle that coincided with Dirac's approach in this case. More generally the $\mathrm{Sp}(2,R)$ principle is extended with spin, supersymmetry and all possible background fields [6][7] and can also apply to p-branes. The $\mathrm{Sp}(2,R)$ point of view has also been crucial to recognize the important consequence of 2T-physics, that there are many gauge choices which lead to many 1T-physics systems, all unified by $\mathrm{Sp}(2,R)$ dualities.

V. EMERGENT $(d-1)+1$ FIELD THEORY IN FLAT SPACETIME GAUGE

Gauge fixing can now be applied either to the action (4.1) or to the equations of motion (4.5). We will now summarize how this is done [14][17] in the “massless relativistic particle” gauge of Table 1. A class of other gauges will be discussed in the next section.

In 2T field theory we do not choose a gauge for X^M like we do for the worldline theory $X^M(\tau)$. But instead we parameterize X^M in a form that is parallel to the various gauge choices in Tables 1,2. Thus, corresponding to the ”massless relativistic particle” gauge of Table 1, we parameterize X^M as follows

$$X^{+'}=\kappa,\ X^{-'}=\kappa\lambda,\ X^{\mu}=\kappa x^{\mu}. \tag{5.1}$$

With this parametrization we see that

$$X^2=\kappa^2\left(-2\lambda+x^2\right), \tag{5.2}$$

$$\left(d^{(d+2)}X\right)\delta\left(X^2\right)=\frac{1}{2}\kappa^{d-1}d\kappa d\lambda d^dx\ \delta\left(\lambda-\frac{x^2}{2}\right). \tag{5.3}$$

Computing the derivatives $\frac{\partial}{\partial X^M}$ of the field $\Phi(X)=\Phi(\kappa,\lambda,x^{\mu})$ via the chain rule gives

$$\frac{\partial\Phi}{\partial X^{\mu}}=\frac{1}{\kappa}\frac{\partial\Phi}{\partial x^{\mu}},\quad\frac{\partial\Phi}{\partial X^{-'}}=\frac{1}{\kappa}\frac{\partial\Phi}{\partial\lambda},\quad\frac{\partial\Phi}{\partial X^{+'}}=\frac{1}{\kappa}\left(\kappa\frac{\partial\Phi}{\partial\kappa}-\lambda\frac{\partial\Phi}{\partial\lambda}-x^{\mu}\frac{\partial\Phi}{\partial x^{\mu}}\right). \tag{5.4}$$

This leads to $X^M\partial_M\Phi(X)=\kappa\frac{\partial}{\partial\kappa}\Phi(\kappa,\lambda,x^{\mu})$ and to the following form of the Laplacian

$$\partial^M\partial_M\Phi=\frac{1}{\kappa^2}\left[\left(\frac{\partial}{\partial x^{\mu}}+x_{\mu}\frac{\partial}{\partial\lambda}\right)^2-\left(2\kappa\frac{\partial}{\partial\kappa}+d-2\right)\frac{\partial}{\partial\lambda}+\left(2\lambda-x^2\right)\left(\frac{\partial}{\partial\lambda}\right)^2\right]\Phi. \tag{5.5}$$

We now solve explicitly the equations that follow from $\delta\left(X^2\right)$ and $\left(X\cdot\partial+\frac{d-2}{2}\right)\Phi(X)=0$, and determine the field configuration that obeys these *kinematic* conditions, leaving the dynamics of Eq.(5.5) for later. This is the quantum analog of solving explicitly two out of the three constraints $X^2=X\cdot P=0$ in the worldline theory at the classical level as displayed in the tables.

We begin by using the 2T-gauge-symmetry which allows us to first choose the gauge in which the remainder $\tilde{\Phi}(X)$ in the field

$$\Phi(X)=\Phi_0(X)+X^2\tilde{\Phi}(X)=\Phi_0(\kappa,x^{\mu})-2\kappa^2\left(\lambda-\frac{x^2}{2}\right)\tilde{\Phi}(\kappa,\lambda,x^{\mu}) \tag{5.6}$$

vanishes with a gauge choice $\tilde{\Phi}(X)=0$. The field $\Phi_0(X)$ is independent of X^2 by definition, and therefore it is also independent of λ (consider the series expansion in powers of $\lambda-\frac{q^2(x)}{2}$).

Hence, in this gauge we have $\frac{\partial\Phi}{\partial\lambda}=0$. This is the analog in the worldline theory to using the gauge $P^{+'}(\tau)=0$ of Table 1, whose quantum equivalent is $P^{+'}\Phi=-i\frac{\partial\Phi}{\partial X^{-\prime}}\ =-i\frac{1}{\kappa}\frac{\partial\Phi}{\partial\lambda}=0$. With this, all dependence on λ in the field has disappeared and the remaining field takes the λ independent form $\Phi(\kappa,x^{\mu})$. Hence λ appears only in the volume element (5.3) and can be integrated out.

Next we solve the kinematic equation $\left(X\cdot\partial+\frac{d-2}{2}\right)\Phi(X)=\left(\kappa\frac{\partial}{\partial\kappa}+\frac{d-2}{2}\right)\Phi(\kappa,\lambda,x^{\mu})=0$, which results in the solution

$$\Phi(\kappa,x^{\mu})=\kappa^{-\frac{d-2}{2}}\phi(x^{\mu}) \tag{5.7}$$

with an arbitrary $\phi(x)$. Inserting this into the action (4.1), and using the now greatly simplified Laplacian

$$\partial^M\partial_M\Phi=\kappa^{-\frac{d+2}{2}}\frac{\partial^2\phi(x)}{\partial x^{\mu}\partial x_{\mu}}, \tag{5.8}$$

we obtain a reduced action in $(d-1)+1$ dimensions for the interacting field $\phi(x^{\mu})$

$$S(\Phi)=Z\int\left(\kappa^{d+1}d\kappa d\lambda d^dx\right)\frac{1}{2\kappa^2}\delta\left(\lambda-\frac{x^2}{2}\right) \tag{5.9}$$

$$\times\left[\frac{1}{2}\left(\kappa^{-\frac{d-2}{2}}\phi\right)\left(\kappa^{-\frac{d+2}{2}}\frac{\partial^2\phi}{\partial x^{\mu}\partial x_{\mu}}\right)-\gamma\frac{d-2}{2d}\left(\kappa^{-\frac{d-2}{2}}\phi\right)^{\frac{2d}{d-2}}\right],$$

$$=\left(Z\int\frac{d\kappa}{2\kappa}\right)\int d^dx\left[\frac{1}{2}\phi\frac{\partial^2\phi}{\partial x^{\mu}\partial x_{\mu}}-\gamma\frac{d-2}{2d}\phi^{\frac{2d}{d-2}}\right], \tag{5.10}$$

The overall factor is normalized by choosing $Z\int\frac{d\kappa}{2\kappa}=1$. We then obtain the reduced action after an integration by parts

$$S(\phi)=\int d^dx\left[-\frac{1}{2}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\phi}{\partial x_{\mu}}-\gamma\frac{d-2}{2d}\phi^{\frac{2d}{d-2}}\right]. \tag{5.11}$$

From here we can proceed to derive the equation of motion for $\phi(x)$ from this action or from the original field equation (4.5) with equivalent results.

The 1T-physics interpretation of the emergent action (5.11) is the standard Klein-Gordon massless scalar with a scale invariant $\phi^{\frac{2d}{d-2}}$ interaction. It is well known (at least in $d=4$, ϕ^4 theory) that this action has a conformal SO$(d,2)$ symmetry at the classical field theory level. Clearly this SO$(d,2)$ hidden conformal symmetry [22] is nothing but the manifest Lorentz symmetry of the original action (4.1), and is one of the indications that the underlying theory is indeed a theory in $d+2$ dimensions.

By studying other gauges, such as those listed in Tables 1,2, it is clear that this message of $d+2$ dimensions with a hidden SO$(d,2)$ symmetry (whose interpretation is different from

conformal symmetry) will be repeated in any other gauge. Furthermore, the emergent field theories in different gauges will correspond to different 1T-physics interpretations of the same $d+2$ system (4.5), but with predicted field theoretic duality relations among themselves. This is one of the non-trivial outputs of 2T-physics, as explored partially in the following sections, for which 1T-physics gives no clues at all.

VI. EMERGENT $(d-1)+1$ FIELD THEORY IN CURVED SPACETIME GAUGES

We will now discuss a family of gauge choices of the 2T-physics field theory (4.1) leading to the following Klein-Gordon field theory for an interacting scalar field in various curved spacetime backgrounds

$$S(\phi, g) = \int d^d x \, \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu}(x) \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} - \frac{d-2}{8(d-1)} R(g) \phi^2 - \gamma \frac{d-2}{2d} \phi^{\frac{2d}{d-2}} \right]. \tag{6.1}$$

The explicit form of $R(g)$ for all the metrics listed in (2.1) are given in Eq.(3.12). The special coefficient in front of the curvature term $R(g)$ indicates that this system is recognized as the conformal scalar in some special backgrounds. Its properties will be discussed in the following section. The resulting class of special metrics $g_{\mu\nu}(x)$ consists of all possible conformally flat metrics that can be written as in Eq.(3.8). Among these we note some interesting cases including those given in the list (2.1). In particular, when the spacetime is such that $R(g)$ is a constant (example $\mathrm{AdS}_{d-n} \times \mathrm{S}^n$, maximally symmetric space, etc.), then that term is similar to a mass term. We will derive $S(\phi, g)$ for all $g_{\mu\nu}$ in our list (2.1) by starting from the 2T field theory (4.1) and treating the field theoretic equivalent of the general conformal gauge given in Table 1 for the worldline theory.

This family of field theoretic gauges does not include the gauge choices in which some components of $X^M(x,p)$ on the worldline depend both on x^μ and p^μ, as seen for some items listed in Tables 1,2, including the massive relativistic, non-relativistic, H-atom gauges, and more. A field theoretic study of the non-relativistic massive particle gauge can be found in [14]. We will return to gauges with nontrivial momentum dependence in $X^M(x,p)$ in future research with the aim of studying a possible origin for mass in the 2T-physics context.

In this paper we will see that there is hidden $\mathrm{SO}(d,2)$ invariance in each curved space theory under a non-linear action of a deformed conformal group as a manifestation of the

original global SO$(d,2)$ symmetry of the parent theory (4.1). Furthermore, the different emergent field theories in $(d-1)+1$ dimensions distinguished by their background metrics will be related to each other by the explicit duality transformations given in the next section.

To derive these claims, we work in the light cone basis, $M=(+',-',m)$, and parameterize $X^M(\kappa,\lambda,x^\mu)$ as in (5.1), in parallel to the general conformal gauge in Table 1. This has the following form

$$X^M(\kappa,\lambda,x^\mu)=\kappa e^{\sigma(x)}\left(\overset{+'}{1},\ \overset{-'}{\lambda},\ \overset{m}{q^m(x)}\right) \tag{6.2}$$

Solving for κ, λ and $q^m(x)$, in terms of $X^{\pm'},X^m$ we get the inverse parametrization

$$\kappa=e^{-\sigma(x)}X^{+'},\quad \lambda=\frac{X^{-'}}{X^{+'}},\quad q^m(x)=\frac{X^m}{X^{+'}}. \tag{6.3}$$

From $q^m(x)=\frac{X^m}{X^{+'}}$ we solve for $x^\mu=f^\mu\left(\frac{X^m}{X^{+'}}\right)$, where $f^\mu(q^m)$ is the inverse map of $q^m(x^\mu)$. This inverse map is inserted in $\sigma(x)=\sigma\left(f^\mu\left(\frac{X^m}{X^{+'}}\right)\right)$ in Eq.(6.3) to complete the full solution of $\kappa=X^{+'}\exp\left(-\sigma\left(f^\mu\left(\frac{X^m}{X^{+'}}\right)\right)\right)$ in terms of $X^{\pm'},X^m$. The field $\Phi(X)=\Phi(\kappa,\lambda,x^\mu)$ is now considered a function of κ,λ,x^μ. The $(d-1)+1$ spacetime x^μ has been embedded in $d+2$ dimensions in different forms that vary as the functions $q^m(x)$ and $\sigma(x)$ change (see examples in Tables 1,2).

With this parametrization we see that

$$X^2=(\kappa e^\sigma)^2\left(-2\lambda+q^2(x)\right), \tag{6.4}$$

$$\left(d^{(d+2)}X\right)\delta\left(X^2\right)=\frac{1}{2}\kappa^{d-1}\det\left(e_\mu^m(x)\right)\ d\kappa d\lambda d^dx\ \delta\left(\lambda-\frac{q^2(x)}{2}\right). \tag{6.5}$$

Here we have taken into account the Jacobian

$$J\left(\frac{X^{+'},X^{-'},X^m}{\kappa,\lambda,x^\mu}\right)=\kappa^{d+1}e^{(d+2)\sigma}\det\left(\partial_\mu q^m\right)=\kappa^{d+1}e^{2\sigma}\det\left(e_\mu^m(x)\right), \tag{6.6}$$

where the vielbein

$$e_\mu^m(x)=e^{\sigma(x)}\frac{\partial q^m(x)}{\partial x^\mu} \tag{6.7}$$

is the same as the one that emerged in the worldline theory in Eq.(3.8).

Using the chain rule, we can then compute the partial derivatives $\frac{\partial\Phi}{\partial X^M}$ in terms of $\frac{\partial\Phi}{\partial\kappa},\frac{\partial\Phi}{\partial\lambda},\frac{\partial\Phi}{\partial x^\mu}$. The result is

$$\frac{\partial\Phi}{\partial X^{-'}}=\frac{1}{\kappa}e^{-\sigma}\frac{\partial\Phi}{\partial\lambda} \tag{6.8}$$

$$\frac{\partial\Phi}{\partial X^m}=\frac{1}{\kappa}\left(-e_m^\mu\partial_\mu\sigma\ \kappa\frac{\partial\Phi}{\partial\kappa}+e_m^\mu\partial_\mu\Phi\right) \tag{6.9}$$

$$\frac{\partial\Phi}{\partial X^{+'}}=\frac{1}{\kappa}\left(\left[e^{-\sigma}+q^m e_m^\mu\partial_\mu\sigma\right]\ \kappa\frac{\partial\Phi}{\partial\kappa}-e^{-\sigma}\lambda\frac{\partial\Phi}{\partial\lambda}-q^m e_m^\mu\partial_\mu\Phi\right) \tag{6.10}$$

Here $e_m^\mu(x)$ is the inverse vielbein which can also be written as $e_m^\mu(x) = e^{-\sigma(x)}\frac{\partial x^\mu}{\partial q^m} = e^{-\sigma(x)}\frac{\partial f^\mu(q)}{\partial q^m}(x)$, where $x^\mu = f^\mu(q)$ is the inverse map discussed following Eq.(6.3). This is verified by using the chain rule $e_\nu{}^m(x)\, e_m^\mu(x) = e^{\sigma(x)}\frac{\partial q^m}{\partial x^\nu}e^{-\sigma(x)}\frac{\partial x^\mu}{\partial q^m} = \frac{\partial x^\mu}{\partial x^\nu} = \delta_\nu{}^\mu$. We note that the dimension operator takes a simple form $\kappa\frac{\partial}{\partial\kappa}$

$$X\cdot\partial\Phi = -X^{+'}\frac{\partial\Phi}{\partial X^{-'}} - X^{-'}\frac{\partial\Phi}{\partial X^{+'}} + X^m\frac{\partial\Phi}{\partial X^m} = \kappa\frac{\partial\Phi}{\partial\kappa}. \tag{6.11}$$

We are now ready to choose gauges and solve the two kinematic equations following in the footsteps of the previous section. We begin by using the 2T-gauge-symmetry which allows us to first choose the gauge in which the remainder $\tilde\Phi(X)$ in the field

$$\Phi(X) = \Phi_0(X) + X^2\tilde\Phi(X) = \Phi_0(\kappa, x^\mu) - 2(\kappa e^\sigma)^2\left(\lambda - \frac{q^2(x)}{2}\right)\tilde\Phi(\kappa,\lambda,x^\mu) \tag{6.12}$$

vanishes with a gauge choice $\tilde\Phi(X) = 0$. The field $\Phi_0(X)$ is independent of X^2 by definition, and therefore it is also independent of λ (consider the series expansion in powers of $\lambda - \frac{q^2(x)}{2}$). Hence, in this gauge we have $\frac{\partial\Phi}{\partial\lambda} = 0$ so that the remaining field takes the λ independent form $\Phi(\kappa, x^\mu)$ everywhere in the action (4.1). Hence λ appears only in the volume element (6.5) and can be integrated out. This is the analog in the worldline theory to using the gauge $P^{+'}(\tau) = 0$ of Table 1, whose quantum equivalent is $P^{+'}\Phi = -i\frac{\partial\Phi}{\partial X^{-'}} = -i\frac{1}{\kappa}e^{-\sigma}\frac{\partial\Phi}{\partial\lambda} = 0$. Next we solve the kinematic equation in Eq.(4.5) $\left(X\cdot\partial + \frac{d-2}{2}\right)\Phi(X) = \left(\kappa\frac{\partial}{\partial\kappa} + \frac{d-2}{2}\right)\Phi(\kappa,\lambda,x^\mu) = 0$, which results in the solution

$$\Phi = \kappa^{-\frac{d-2}{2}}\phi(x^\mu). \tag{6.13}$$

So far we have reduced the field to $(d-1)+1$ dimensions by solving the kinematic constraints in a convenient gauge. We are now ready to analyze the dynamics satisfied by the field $\phi(x^\mu)$. We compute the Laplacian $\partial^M\partial_M\Phi = -2\frac{\partial}{\partial X^{+'}}\frac{\partial}{\partial X^{-'}}\Phi + \eta^{mn}\frac{\partial}{\partial X^m}\frac{\partial}{\partial X^n}\Phi$ by using Eqs.(6.8-6.10). Recalling to drop all terms containing $\frac{\partial\Phi}{\partial\lambda} = 0$, we have

$$\begin{aligned}\partial^M\partial_M\Phi &= 0 + \eta^{mn}\frac{\partial}{\partial X^m}\frac{\partial}{\partial X^n}\Phi \\ &= \eta^{mn}\left(-e_m^\mu\partial_\mu\sigma\,\frac{\partial}{\partial\kappa} + \frac{1}{\kappa}e_m^\mu\partial_\mu\right)\left(-e_n^\nu\partial_\nu\sigma\,\frac{\partial\Phi}{\partial\kappa} + \frac{1}{\kappa}e_n^\nu\partial_\nu\Phi\right)\end{aligned} \tag{6.14}$$

After inserting $\Phi(\kappa, x^\mu) = \kappa^{-\frac{d-2}{2}}\phi(x^\mu)$ and some calculation, this takes the form

$$\partial^M\partial_M\Phi = \kappa^{-\frac{d+2}{2}}\left\{\begin{array}{c} g^{\mu\nu}\partial_\mu\partial_\nu\phi + \left(e^{n\mu}\partial_\mu e_n^\nu + (d-1)\,g^{\mu\nu}\partial_\mu\sigma\right)\partial_\nu\phi \\ +\left[\frac{(d-2)d}{4}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \frac{d-2}{2}\left(g^{\mu\nu}\partial_\mu\partial_\nu\sigma + e^{n\mu}\partial_\mu e_n^\nu\partial_\nu\sigma\right)\right]\phi \end{array}\right\} \tag{6.15}$$

where we have defined the metric

$$g^{\mu\nu}(x) \equiv \eta^{mn} e_m^{\mu}(x) e_n^{\nu}(x) = e^{2\sigma(x)} \eta_{mn} \partial_\mu q^m(x) \partial_\nu q^n(x), \tag{6.16}$$

This metric is conformally flat. The expression can be further simplified to take the form of the Laplacian operator for the metric $g^{\mu\nu}(x)$ with an additional curvature term[9]

$$\partial^M \partial_M \Phi = \kappa^{-\frac{d+2}{2}} \left\{ \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) - \frac{d-2}{4(d-1)} R(g) \phi \right\}. \tag{6.18}$$

The curvature scalar $R(g)$ for the given metric in Eq.(6.16) is computed in Appendix A

$$R(g) = (1-d) \left[d g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + 2 g^{\mu\nu} \partial_\mu \partial_\nu \sigma + 2 e^{n\mu} \partial_\mu e_n^{\nu} \partial_\nu \sigma \right] \tag{6.19}$$

Inserting the reduced forms for the field (6.13), the Laplacian (6.18), and the volume element (6.5) into the $d+2$ dimensional action (4.1), and repeating the steps of the reduction procedure similar to Eq.(5.9), we finally derive the emergent action for the interacting conformal scalar given in Eq.(6.1).

In the following two sections it will be shown that two such emergent actions, with different background metrics $g_{\mu\nu}(x)$ and $\tilde{g}_{\mu\nu}(x)$ such as those listed in (2.1), are dual to each other, while for each fixed metric $g_{\mu\nu}(x)$ of this type the action is invariant under a hidden SO$(d,2)$ global symmetry. These properties emanate, of course, from the the original action, and are indications of the underlying spacetime in $d+2$ dimensions.

VII. DUALITIES

Given any two metrics $g_{\mu\nu}(x)$ and $\tilde{g}_{\mu\nu}(x)$ in $(d-1)+1$ dimensions, built from $(\sigma(x^\mu), q^m(x^\mu))$ or $(\tilde{\sigma}(x^\mu), \tilde{q}^m(x^\mu))$ respectively as in Eq.(6.16), we have two KG field theories that are considered theories in different fixed background spacetimes from the point of view of 1T-physics. However, since we obtained them by gauge fixing the same parent

[9] To bring the Laplacian to this form we have used the following identities $\sqrt{-g} = e = \det\left(e_\mu^m\right)$, and $\partial_\mu g^{\mu\nu} = \partial_\mu \left(e^{n\mu} e_n^{\nu}\right) = e^{n\mu} \partial_\mu e_n^{\nu} + e_n^{\nu} \partial_\mu e^{n\mu}$, and

$$\frac{1}{e} \partial^\nu e = \partial^\nu \ln\left[\det\left(e_\mu^k\right)\right] = \partial^\nu \ln\left[e^{tr \ln\left(e_\mu^k\right)}\right] = tr\left[\partial^\nu \ln\left(e_\mu^k\right)\right] = tr\left[e_k^\lambda \partial^\nu e_\mu^k\right] = e_k^\mu \partial^\nu e_\mu^k \tag{6.17}$$

which gives $\frac{1}{e} \partial^\nu e + e_n^\nu \partial_\mu e^{n\mu} = e_k^\mu \partial^\nu e_\mu^k + e_n^\nu \partial_\mu e^{n\mu} = (d-1) \partial^\nu \sigma$.

2T field theory $S(\Phi)$ in $d+2$ dimensions we have the 2T-physics prediction that they are in some sense the same action

$$S(\phi, g_{\mu\nu}) = S(\Phi) = S\left(\tilde{\phi}, \tilde{g}_{\mu\nu}\right). \tag{7.1}$$

So we expect a duality transformation that relates two different 1T-physics actions $S(\phi, g_{\mu\nu})$, $S\left(\tilde{\phi}, \tilde{g}_{\mu\nu}\right)$, for the classes of *metrics related by 2T-physics* as specified in the previous section. This duality transformation is constructed explicitly in this section.

In the worldline theory at the classical level the relationship between any two gauges in Tables 1,2 is given by an Sp$(2,R)$ gauge transformation in *phase space*

$$\begin{pmatrix} \tilde{X}^M\left(\tilde{t}, \tilde{H}, \tilde{\mathbf{r}}, \tilde{\mathbf{p}}\right) \\ \tilde{P}_M\left(\tilde{t}, \tilde{H}, \tilde{\mathbf{r}}, \tilde{\mathbf{p}}\right) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X^M(t, H, \mathbf{r}, \mathbf{p}) \\ P_M(t, H, \mathbf{r}, \mathbf{p}) \end{pmatrix} \tag{7.2}$$

where the Sp$(2,R)$ parameters $(a,b,c,d)(\tau)$ that are local on the worldline can be written as functions of either set of canonical variables $\left(\tilde{t}, \tilde{H}, \tilde{\mathbf{r}}, \tilde{\mathbf{p}}\right)(\tau)$ or $(t, H, \mathbf{r}, \mathbf{p})(\tau)$. What form does this transformation take in the local field theory in *position space* alone, and in terms of the dynamical field $\phi(x)$?

For the class of spacetimes we have specialized above the $X^M(x^\mu)$ is a function of position space only (does not contain p^μ), and therefore the Sp$(2,R)$ gauge transformation (7.2) is easily rephrased as a transformation of the type

$$(\tilde{\sigma}(x^\mu), \tilde{q}^m(x^\mu)) \to (\sigma(x^\mu), q^m(x^\mu)) \tag{7.3}$$

among the metrics (6.16) such as those listed in (2.1). This amounts to general coordinate transformations $x^\mu \to y^\mu(x)$ and local Weyl rescaling of the metric, which we will implement below in Eqs.(7.4-7.7) in the field theory language. It is evident that our approach allows us to contemplate the more general duality transformation that goes beyond (7.3) and thereby include in our discussion field theoretic duality transformations to the other gauge types $X^M(t, H, \mathbf{r}, \mathbf{p})$ listed in Tables 1,2, including the massive relativistic, massive non-relativistic, and H-atom gauges. However, for simplicity we concentrate in this paper on the easier case of type (7.3).

The action (6.1) is formally invariant under general coordinate transformations, but of course, since the metric is not dynamical, the action is not actually invariant. Instead, the general coordinate transformation $x^\mu \to y^\mu(x)$ of the dynamical field $\phi(x)$ maps the theory

into another theory with a new background metric that is related to the old one by the following transformations

$$\phi\left(x\right)\rightarrow\tilde{\phi}\left(x\right)=\phi\left(y\left(x\right)\right), \tag{7.4}$$

$$g_{\mu\nu}\left(x\right)\rightarrow\tilde{g}_{\mu\nu}\left(x\right)=\partial_{\mu}y^{\lambda}\left(x\right)\partial_{\nu}y^{\sigma}\left(x\right)g_{\lambda\sigma}\left(y\left(x\right)\right). \tag{7.5}$$

Less familiar is that, thanks to special coefficient of the curvature term, the action (6.1) is also formally invariant under the following Weyl rescaling [24] of the field ϕ and metric $g_{\mu\nu}$

$$\phi\left(x\right)\rightarrow\tilde{\phi}\left(x\right)=e^{-\frac{d-2}{2}\lambda(x)}\phi\left(x\right), \tag{7.6}$$

$$g_{\mu\nu}\left(x\right)\rightarrow\tilde{g}_{\mu\nu}\left(x\right)=e^{2\lambda(x)}g_{\mu\nu}\left(x\right), \tag{7.7}$$

as proven below. The rescaling and general coordinate transformations of the field ϕ are induced through the expression $\Phi\left(X\right)=\kappa^{-\frac{d-2}{2}}\phi\left(x\right)$ in Eq.(6.13), and can be understood as originating from the Sp$(2,R)$ gauge transformation (7.2,7.3) of the variables (κ,λ,x^{μ}) defined through (6.3).

Again it should be emphasized that this Weyl rescaling is not an invariance of the action (6.1) since the metric $g_{\mu\nu}\left(x\right)$ is not dynamical. Rather, this is a duality transformation to another theory with a new background metric $\tilde{g}_{\mu\nu}\left(x\right)$.

The duality under the general coordinate transformation is evident. We will now prove the duality under the Weyl rescaling (7.6,7.7). For this, it will be useful to provide the following transformation rules for the curvature tensors which are well known

$$\tilde{\Gamma}_{\mu\nu}^{\lambda}=\Gamma_{\mu\nu}^{\lambda}+\delta_{\mu}^{\lambda}\partial_{\nu}\lambda+\delta_{\nu}^{\lambda}\partial_{\mu}\lambda-g_{\mu\nu}g^{\lambda\kappa}\partial_{\kappa}\lambda, \tag{7.8}$$

$$\tilde{R}^{\lambda}{}_{\kappa\mu\nu}=R^{\lambda}{}_{\kappa\mu\nu}-2\left\{\begin{array}{c}\left(\delta_{[\mu}^{\lambda}\delta_{\nu]}^{\alpha}\delta_{\kappa}^{\beta}-\tilde{g}_{\kappa[\mu}\delta_{\nu]}^{\alpha}\tilde{g}^{\lambda\beta}\right)\nabla_{\alpha}\left(\partial_{\beta}\lambda\right)\\-\left(\delta_{[\mu}^{\lambda}\delta_{\nu]}^{\alpha}\delta_{\kappa}^{\beta}-\tilde{g}_{\kappa[\mu}\delta_{\nu]}^{\alpha}\tilde{g}^{\lambda\beta}+\tilde{g}_{\kappa[\mu}\delta_{\nu]}^{\lambda}\tilde{g}^{\alpha\beta}\right)\partial_{\alpha}\lambda\partial_{\beta}\lambda\end{array}\right\}, \tag{7.9}$$

$$\tilde{R}_{\mu\nu}=R_{\mu\nu}-\left[\left(d-2\right)\delta_{\mu}^{\kappa}\delta_{\nu}^{\lambda}+g_{\mu\nu}g^{\kappa\lambda}\right]\nabla_{\kappa}\left(\partial_{\lambda}\lambda\right)+\left(d-2\right)\left(\delta_{\mu}^{\kappa}\delta_{\nu}^{\lambda}-g_{\mu\nu}g^{\kappa\lambda}\right)\partial_{\kappa}\lambda\partial_{\lambda}\lambda, \tag{7.10}$$

$$\tilde{R}=e^{-2\lambda}\left\{R-\left(d-1\right)\left[2g^{\mu\nu}\nabla_{\mu}\left(\partial_{\nu}\lambda\right)+\left(d-2\right)g^{\mu\nu}\partial_{\mu}\lambda\partial_{\nu}\lambda\right]\right\}, \tag{7.11}$$

where ∇_{μ} is the general covariant derivative. It is now easy to check that the Lagrangian $\mathcal{L}$ is invariant up to a total divergence. First we substitute for $\tilde{\phi},\tilde{g}_{\mu\nu},\tilde{R}$

$$\tilde{\mathcal{L}}=e^{\lambda d}\sqrt{-g}\left(\begin{array}{c}-\frac{1}{2}\left(e^{-2\lambda}g^{\mu\nu}\right)\partial_{\mu}\left(e^{-\frac{d-2}{2}\lambda}\phi\right)\partial_{\nu}\left(e^{-\frac{d-2}{2}\lambda}\phi\right)-\gamma\frac{d-2}{2d}e^{-d\lambda}\phi^{\frac{2d}{d-2}}\\-\frac{d-2}{8(d-1)}e^{-2\lambda}\left\{R-\left(d-1\right)\left[2g^{\mu\nu}\nabla_{\mu}\left(\partial_{\nu}\lambda\right)+\left(d-2\right)g^{\mu\nu}\partial_{\mu}\lambda\partial_{\nu}\lambda\right]\right\}e^{-(d-2)\lambda}\phi^{2}\end{array}\right) \tag{7.12}$$

For infinitesimal $\lambda(x)$ this expression becomes $\tilde{\mathcal{L}} = \mathcal{L} + \delta_\lambda \mathcal{L}$, where

$$\delta_\lambda \mathcal{L} = \partial_\mu \left(\frac{(d-2)}{4} \sqrt{-g} g^{\mu\nu} \partial_\nu \lambda \phi^2 \right). \tag{7.13}$$

Since $\delta_\lambda \mathcal{L}$ is a total divergence, this proves that the two actions $S(g,\phi)$ and $S\left(\tilde{g},\tilde{\phi}\right)$ are indeed related to each other by the duality transformation (7.6,7.7).

Technically, what we have is a family of actions, some of which are listed in (2.1), that are related by coordinate and Weyl transformations of the background metrics, and field redefinitions of the dynamical field $\phi(x)$. This corresponds precisely to the duality predicted by 2T physics in Eq.(7.1). Alternatively, these actions in $(d-1)+1$ dimensions can be taken to be different parameterizations of the same 2T-physics system (4.1) in $d+2$ dimensions, however, these different descriptions do not have the same 1T-physics spacetimes, and therefore they have different 1T-physics interpretations.

VIII. SO$(d,2)$ GLOBAL SYMMETRY AND ITS GENERATORS

In this section, we will show that the KG field theory of Eq.(6.1) has a hidden global SO$(d,2)$ in any of the emergent background spacetimes such as those given in Eq.(3.12), and we derive explicitly the form of the L^{MN} generators.

The original master action (4.1) that led to the emergent field theories (6.1) is manifestly invariant under SO$(d,2)$ global transformations given by

$$\delta_\omega \Phi(X) = \frac{i}{2}\omega_{MN} L^{MN}\Phi(X), \quad L^{MN}\Phi(X) = -i\left(X^M \frac{\partial \Phi(X)}{\partial X^N} - X^N \frac{\partial \Phi(X)}{\partial X^M} \right) \tag{8.1}$$

The form of the L^{MN} in the emergent field theories can be obtained as differential operators from this expression by substituting the parametrization $X^M(\kappa,\lambda,x^\mu) = \kappa e^{\sigma(x)}(\overset{+'}{1},\ \overset{-'}{\lambda},\ \overset{m}{q^m}(x))$ and using the chain rules Eqs.(6.8, 6.9, 6.10). Applying these on the gauge fixed form of the field $\Phi(X) = \kappa^{-\frac{d-2}{2}}\phi(x^\mu)$ given in (6.13), this procedure provides an expression of the form

$$\delta_\omega \Phi(X) = \kappa^{-\frac{d-2}{2}} \frac{i}{2}\omega_{MN}\left(L^{MN}\phi(x^\mu)\right) \equiv \kappa^{-\frac{d-2}{2}}\delta_\omega\phi(x^\mu) \tag{8.2}$$

where $\delta_\omega\phi(x^\mu) = \frac{i}{2}\omega_{MN}L^{MN}\phi(x^\mu)$ is now given in terms of a non-linear differential operator representation of the SO$(d,2)$ generators written in terms of the emergent spacetime in

$(d-1)+1$ dimensions. Since the original action is invariant under $\mathrm{SO}(d,2)$, the emergent action must have a hidden $\mathrm{SO}(d,2)$ symmetry under this transformation.

We now give the explicit expression for $L^{MN}\phi(x^\mu)$. Rather than presenting the result of the straightforward computation we have just outlined above, we provide additional insight by also giving the following arguments based on dualities which lead to the same explicit form for $L^{MN}\phi(x^\mu)$.

Another way to find the generators $L^{MN}\phi(x^\mu)$ is to relate our conformal KG field $\phi(x)$ to the flat KG field $\phi_0(x)$ by the duality transformation in which we take $\lambda(x)=\sigma(x)$ and $y^m(x)=q^m(x)$

$$\phi(x)=e^{-\frac{d-2}{2}\sigma}\phi_0(q(x)). \tag{8.3}$$

Given that the metric in flat space is η_{mn}, the combined duality transformations (7.5,7.7) generate the metric $g_{\mu\nu}=e^{2\sigma}\partial_\mu q^m\partial_\nu q^n\eta_{mn}$.

We can then relate the L^{MN}'s in curved space to the ones in flat space L_0^{MN}

$$\text{flat: } \delta_\omega\phi_0(q)=\frac{i}{2}\omega_{MN}L_0^{MN}\phi_0 \tag{8.4}$$

where the L_0^{MN} in flat space were obtained directly from 2T-physics in [1] as the $L^{MN}=X^MP^N-X^NP^M$ specialized to flat spacetime at the quantum level, and given as

$$L_0^{+\prime m}=p^m,\quad L_0^{+\prime-\prime}=\frac{1}{2}(p_mq^m+q^mp_m)+i \tag{8.5}$$

$$L_0^{-\prime m}=\frac{1}{2}q_kp^mq^k-\frac{1}{2}q^mp_kq^k-\frac{1}{2}q^kp_kq^m-iq^m \tag{8.6}$$

$$L_0^{mn}=q^mp^n-q^np^m \tag{8.7}$$

These correspond to the well known conformal transformations of a scalar field in flat spacetime. They are Hermitian relative to the norm defined for the field $\phi_0(x)$ in relativistic field theory [1]. They close under commutation by using $[q_m,p^n]=i$ and automatically give a constant value for the quadratic Casimir eigenvalue $C_2=\frac{1}{2}L^{MN}L_{MN}=1-d^4/4$ independent of q,p, which corresponds to the singleton.

In the present case, as applied on $\phi_0(q(x))$, we must replace the symbols $q^m, p_m=-i\frac{\partial}{\partial q^m}$ (and $p^m\equiv\eta^{mn}p_n$ and $q_k\equiv\eta_{km}q^m$) above with the expressions $q^m(x)$ and $p_m=-i\frac{\partial x^\mu}{\partial q^m}\partial_\mu$ which can be written as (see comments following (6.10))

$$p_m=-ie^{\sigma(x)}e_m^{\ \mu}(x)\frac{\partial}{\partial x^\mu}. \tag{8.8}$$

These generate a differential operator representation for L_0^{MN} in x^μ rather than q^m space.

For the general case we can now write

$$\delta_\omega \phi(x) = e^{-\frac{d-2}{2}\sigma(x)}\delta_\omega \phi_0(q(x)) = \frac{i}{2}\omega_{MN} e^{-\frac{d-2}{2}\sigma(x)} L_0^{MN} \phi_0(q(x)) \tag{8.9}$$

$$= \frac{i}{2}\omega_{MN} e^{-\frac{d-2}{2}\sigma(x)} L_0^{MN}\left(e^{\frac{d-2}{2}\sigma(x)}\phi(x)\right) \equiv \frac{i}{2}\omega_{MN} L^{MN}\phi(x) \tag{8.10}$$

Hence the general L^{MN} is given by the differential operators

$$L^{MN} = e^{-\frac{d-2}{2}\sigma} L_0^{MN} e^{\frac{d-2}{2}\sigma} \tag{8.11}$$

The closure of the Lie algebra is evident from the form of L^{MN} as a similarity transformation and the known closure of L_0^{MN} as SO$(d,2)$ generators.

In this form, the generator L^{MN} is presented as a combination of duality plus conformal transformations in ordinary flat spacetime. That is, we start with the field $\phi(x)$, transform it via duality (Weyl and general coordinate transformation) to the field in flat space $\phi_0(q)$, apply ordinary SO$(d,2)$ conformal transformations in the flat space q^m, and then apply a duality transformation back to the field $\phi(x)$.

The parameters in this transformation are only the global parameters of SO$(d,2)$ in flat spacetime, while here $\sigma(x)$ and $q^m(x)$ are not parameters since they define the metric $g_{\mu\nu}(x)$. We emphasize that this deformed conformal transformation generates a global SO$(d,2)$ symmetry of the action $S(g,\phi)$ for each *fixed metric* $g_{\mu\nu}(x)$.

Performing the SO$(d,2)$ transformation $\delta_\omega\phi(x) = \frac{i}{2}\omega_{MN}L^{MN}\phi(x)$ one can now see that $\delta_\omega S(g,\phi) = 0$, since the invariance is true for the flat theory with the Minkowski metric $\eta_{\mu\nu}$ and we have also shown that the duality transformation relates the flat and curved theories. Without reference to the flat theory, but only using the generator L^{MN} above, we see that this is a true invariance of the action $S(g,\phi)$ since the fixed background metric $g_{\mu\nu}(x)$ is left unchanged by the SO$(d,2)$ transformation. This hidden global symmetry is nothing but the original global SO$(d,2)$ symmetry of the action $S(\Phi)$ in $d+2$ dimensions, and hence it is one of the indications within 1T-physics of the higher dimensional nature of the underlying system.

IX. CONCLUSION

In this paper, we have shown that the conformal scalar propagating in any conformally flat metric in $(d-1)+1$ dimensions can be obtained using 2T field theory in flat $d+2$

dimensions. The SO$(d,2)$ global symmetry of the 2T theory is realized as the hidden non-linear invariance of the resulting KG equation under the action of a deformed conformal group. The duality between the different conformal KG equations in different backgrounds is a first step to demonstrate the use of duality in 1T-physics as emergent from 2T field theory. As mentioned in the introduction, the availability of such dualities is expected to be an important tool in the study of more complicated cases.

The obvious next step in our investigation is to generalize this paper's results to the spin-$\frac{1}{2}$ and spin-1 cases, and apply these duality ideas to the Standard Model as a theory that emerges from 4+2 dimensions [17]. This is rather straightforward and is done in a companion paper [25].

The general setup of 2T physics presented in this paper teaches us that the particular class of dual theories studied in this paper (which we related to well known properties of the conformal scalar) is only the most evident sector of a much larger duality, which would be much harder to notice and, arguably, impossible to investigate systematically from a strict 1T perspective. This paper is indeed the first step of such a program.

The larger set of dualities already uncovered in the worldline formalism leads us to expect a similar variety in field theory. Of particular interest will be the extension of the theory to allow gauge choices equivalent to those in the worldline formalism which involve mixing of x and p (cf. section III). This may result in dualities between local and non-local field theories at least in some instances. It is to be noted that the appearance of mass in the worldline formalism was related to such gauge choices. This suggests the possibility that mass in field theory might come as a modulus in the embedding of 3+1 dimensional phase space into 4+2 dimensional phase space.

As in other instances of dualities, in principle the class of dualities described in this paper, and the more general dualities provided by 2T-physics can be used as new tools to investigate the properties of the Standard Model, including QCD. For instance, one could use one form of the 1T-physics action to learn some non-perturbative information about the other 1T-physics action.

Through the dualities, but especially through the parent 2T theory, we obtain a new unification of 1T field theories through higher dimensions. This is quite different from the Kaluza-Klein type ideas since there are no Kaluza-Klein remnants either in the form of extra fields or in the form of extra quantum numbers. Instead what we have is a family of dual

theories with a different set of parameters described as the moduli of the metrics (and more generally masses, couplings, etc.), as seen in Tables 1,2.

Further research on these topics is warranted and is currently being pursued.

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APPENDIX A: CURVATURE TENSORS

In this appendix,we provide the Riemann tensor, Ricci tensor, and Ricci scalar of the $(d-1)+1$ spacetime. As a reminder, we use:

$$e_\mu^m \equiv e^\sigma \partial_\mu q^m, g_{\mu\nu} \equiv \eta_{mn} e_\mu^m e_\nu^n = e^{2\sigma} \partial_\mu q^m \partial_\nu q^n \eta_{mn}. \tag{A1}$$

The connection is defined as:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\tau} \left(\partial_\mu g_{\nu\tau} + \partial_\nu g_{\mu\tau} - \partial_\tau g_{\mu\nu}\right) \tag{A2}$$

This gives:

$$\Gamma^\rho_{\mu\nu} = \delta^\rho_\mu \partial_\nu \sigma - g_{\mu\nu} g^{\rho\lambda} \partial_\lambda \sigma + e^\rho_m \partial_\mu e^m_\nu \tag{A3}$$

The Riemann tensor is given by

$$R^\rho{}_{\tau\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\tau} - \partial_\nu \Gamma^\rho_{\mu\tau} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\tau} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\tau} \tag{A4}$$

$$= \begin{bmatrix} \delta^\rho_\mu \partial_\nu\sigma\partial_\tau\sigma - g_{\nu\tau} e^\rho_i \partial_\mu e^{i\lambda}\partial_\lambda\sigma - g_{\nu\tau} g^{\rho\lambda}\partial_\lambda\partial_\mu\sigma \\ -\delta^\rho_\mu g_{\nu\tau} g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma + \delta^\rho_\nu\partial_\tau\sigma\partial_\mu\sigma \\ +\delta^\rho_\nu e_{i\tau}\partial_\mu e^{i\lambda}\partial_\lambda\sigma + \delta^\rho_\nu\partial_\tau\partial_\mu\sigma \end{bmatrix} - (\mu \leftrightarrow \nu) \tag{A5}$$

While the calculation above is straightforward, it is rather tedious and not particularly enlightening. The Ricci tensor is easily obtained: $R_{\tau\nu} = R^\mu{}_{\tau\mu\nu}$

$$\begin{gathered} R_{\tau\nu} = (1-d)\, g_{\nu\tau} g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma \\ -g_{\nu\tau} e^\mu_i \partial_\mu e^{i\lambda}\partial_\lambda\sigma - g_{\nu\tau} g^{\alpha\beta}\partial_\alpha\partial_\beta\sigma \\ (2-d) e_{i\tau}\partial_\nu e^{i\lambda}\partial_\lambda\sigma + (2-d)\partial_\tau\partial_\nu\sigma \end{gathered} \tag{A6}$$

Finally the Ricci scalar is: $R = g^{\tau\nu} R_{\tau\nu}$

$$R = (1-d)\left[d g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + 2g^{\mu\nu}\left(\partial_\mu\partial_\nu\sigma\right) + 2e^{\nu m}\partial_\nu e^\mu_m \partial_\mu\sigma\right] \tag{A7}$$

[1] I. Bars, C. Deliduman and O. Andreev, " Gauged Duality, Conformal Symmetry and Spacetime with Two Times" , Phys. Rev. **D58** (1998) 066004 [arXiv:hep-th/9803188]. For reviews of subsequent work see: I. Bars, " Two-Time Physics" , arXiv:hep-th/9809034; " Survey of two-time physics," arXiv:hep-th/0008164; " 2T-physics 2001," arXiv:hep-th/0106021.

[2] I. Bars and C. Deliduman, Phys. Rev. **D58** (1998) 106004, [arXiv:he:hep-th/9806085.]

[3] I. Bars, "Conformal Symmetry and Duality between Free Particle, H - atom and Harmonic Oscillator", Phys.Rev.**D58** (1998) 066006 [arXiv: hep-th/9804028]. This paper covers the "free relativistic massless particle" and the "hydrogen atom" gauges.

[4] I. Bars, "Hidden Symmetries, AdS_D x S^n, and the Lifting of One-Time-Physics to Two-Time-Physics', Phys.Rev.**D59** (1999) 045019 [arXiv:hep-th/9810025]. This covers the "non-relativistic particle", the "massive relativistic. particle" and the "particle on $\mathrm{AdS}_{d-n} * S^n$" gauges (the $\mathrm{AdS}_{d-n} * S^n$ case uses a slightly different parametrization).

[5] I. Bars and M. Picon, "Single twistor description of massless, massive, AdS, and other interacting particles," Phys. Rev. **D73** (2006) 064002 [arXiv:hep-th/0512091]. The "sphere*time" gauge is presented in the context of 2T twistor theory.

[6] I. Bars, "Two time physics with gravitational and gauge field backgrounds", Phys. Rev. **D62**, 085015 (2000) [arXiv:hep-th/0002140]

[7] I. Bars and C. Deliduman, " High spin gauge fields and two time physics", Phys. Rev. **D64**, 045004 (2001) [arXiv:hep-th/0103042].

[8] I. Bars, " 2T physics formulation of superconformal dynamics relating to twistors and supertwistors," Phys. Lett. B **483**, 248 (2000) [arXiv:hep-th/0004090]. "Twistors and 2T-physics," AIP Conf. Proc. **767** (2005) 3 [arXiv:hep-th/0502065].

[9] I. Bars and M. Picon, "Single twistor description of massless, massive, AdS, and other interacting particles," Phys. Rev. **D73** (2006) 064002 [arXiv:hep-th/0512091]; "Twistor Transform in d Dimensions and a Unifying Role for Twistors," Phys. Rev. **D73** (2006) 064033, [arXiv:hep-th/0512348].

[10] I. Bars, "Lectures on twistors," [arXiv:hep-th/0601091], appeared in *Superstring Theory and M-theory*, Ed. J.X. Lu, page ; and in *Quantum Theory and Symmetries IV*, Ed. V.K. Dobrev, Heron Press (2006), Vol.2, page 487 (Bulgarian Journal of Physics supplement, Vol. 33).

[11] I. Bars, C. Deliduman and D. Minic, "Supersymmetric Two-Time Physics", Phys. Rev. **D59** (1999) 125004, hep-th/9812161; "Lifting M-theory to Two-Time Physics", Phys. Lett. **B457** (1999) 275 [arXiv:he:hep-th/9904063].

[12] I. Bars, "Twistor superstring in 2T-physics," Phys. Rev. **D70** (2004) 104022, [arXiv:hep-th/0407239].

[13] I. Bars, "Twistors and 2T-physics," AIP Conf. Proc. **767** (2005) 3 , [arXiv:hep-th/0502065].

[14] I.Bars, "Two-Time Physics in Field Theory", Phys. Rev. D 62, 046007 (2000), [arXiv:hep-th/0003100].

[15] I. Bars, " Hidden 12-dimensional structures in AdS_5 x S^5 and M^4 x R^6 supergravities," Phys. Rev. D **66**, 105024 (2002) [arXiv:hep-th/0208012]; " A mysterious zero in $AdS_(5)$ x S^5 supergravity," Phys. Rev. D **66**, 105023 (2002) [arXiv:hep-th/0205194].

[16] I. Bars and Y-C. Kuo, "Interacting two-time Physics Field Theory with a BRST gauge Invariant Action", Phys. Rev. **D74 (**2006) 085020, hep-th/0605267.

[17] I. Bars, "The standard model of particles and forces in the framework of 2T-physics", Phys. Rev. **D74** (2006) 085019 [arXiv:hep-th/0606045]. For a summary see "The Standard Model as a 2T-physics theory," arXiv:hep-th/0610187.

[18] I. Bars and Y-C. Kuo, "Field Theory in 2T-physics with $N = 1$ Supersymmetry", hep-th/0702089; "Supersymmetric 2T-physics field theory", arXiv:hep-th/0703002 .

[19] I. Bars and B. Orcal, "Generalized Twistor Transform And Dualities, With A New Description of Particles With Spin, Beyond Free and Massless," arXiv:0704.0296 [hep-th].

[20] I. Bars and S-H. Chen, to be published.

[21] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley and Sons (1972).

[22] P.A.M Dirac, Ann. Math. **37** (1036) 429.

[23] H. A. Kastrup, Phys. Rev. **150** (1966) 1183.

[24] E. S. Fradkin and A. A. Tseytlin, "Conformal Supergravity," Phys. Rept. **119**, 233 (1985).

[25] I. Bars and G. Quélin, to be published.